Formal Analysis: Lecture 5

Raymond Duch

Nuffield College

February 14, 2011

Extensive Form Games: Simultaneous moves

- Games may involve both sequential and simultaneous actions. For example, in elections, candidates position themselves and later voters vote (simultaneously).

Extensive Form Games: Simultaneous moves

- Games may involve both sequential and simultaneous actions.
  For example, in elections, candidates position themselves and later voters vote (simultaneously).
- There are five components of extensive form games with simultaneous moves

Extensive Form Games: Simultaneous moves

- Games may involve both sequential and simultaneous actions.
  For example, in elections, candidates position themselves and later voters vote (simultaneously).
- There are five components of extensive form games with simultaneous moves
  - a set of players

Extensive Form Games: Simultaneous moves

- Games may involve both sequential and simultaneous actions. For example, in elections, candidates position themselves and later voters vote (simultaneously).
- There are five components of extensive form games with simultaneous moves
  - a set of players
  - a set of sequences (terminal histories)

Extensive Form Games: Simultaneous moves

- Games may involve both sequential and simultaneous actions. For example, in elections, candidates position themselves and later voters vote (simultaneously).
- There are five components of extensive form games with simultaneous moves
  - a set of players
  - a set of sequences (terminal histories)
  - a player function that assigns a set of players to every proper subsequence of some terminal history. **Note that now there may be multiple players after a given subsequence**

Extensive Form Games: Simultaneous moves

- Games may involve both sequential and simultaneous actions. For example, in elections, candidates position themselves and later voters vote (simultaneously).
- There are five components of extensive form games with simultaneous moves
  - a set of players
  - a set of sequences (terminal histories)
  - a player function that assigns a set of players to every proper subsequence of some terminal history. **Note that now there may be multiple players after a given subsequence**
  - preferences over the set of terminal histories

Extensive Form Games: Simultaneous moves

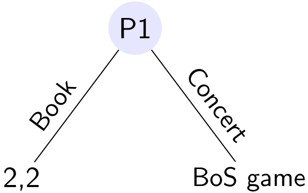
- Games may involve both sequential and simultaneous actions. For example, in elections, candidates position themselves and later voters vote (simultaneously).
- There are five components of extensive form games with simultaneous moves
  - a set of players
  - a set of sequences (terminal histories)
  - a player function that assigns a set of players to every proper subsequence of some terminal history. **Note that now there may be multiple players after a given subsequence**
  - preferences over the set of terminal histories
  - **and** a set of actions $A_i(h)$ for each proper subhistory h and each player i assigned by the player function. **Note that this opens the possibility of several players taking actions simultaneously at subhistory h**

Extensive Form Games: Simultaneous moves

- Games may involve both sequential and simultaneous actions. For example, in elections, candidates position themselves and later voters vote (simultaneously).
- There are five components of extensive form games with simultaneous moves
  - a set of players
  - a set of sequences (terminal histories)
  - a player function that assigns a set of players to every proper subsequence of some terminal history. **Note that now there may be multiple players after a given subsequence**
  - preferences over the set of terminal histories
  - **and** a set of actions $A_i(h)$ for each proper subhistory h and each player i assigned by the player function. **Note that this opens the possibility of several players taking actions simultaneously at subhistory h**
- It is usually hard to graph this type of games. Now game trees must accommodate several players in the same node.

A variant of the Battle of the Sexes

At the beginning of the game, Player 1 has two options: stay home to read a book; or try to coordinate with Player 2 to attend a concert.



If Player 1 chooses Concert, then she plays the BoS game with Player 2.

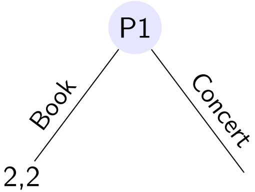
		Player 2	
		B	S
Player 1	B	$3, 1$	$0, 0$
	S	$0, 0$	$1, 3$

Extensive Form with simultaneous moves

Solving extensive games with simultaneous moves

To solve this type of games: What we first need is a complete list of possible strategies for each player.
Strategy: A function that assigns an action to each *history* in which it is the player's turn, *including those histories where other players also have actions to choose from*.
Important: Now a strategy must also specify what a player will do in subhistories where the player function is assigning other players as well
In other words: A strategy is a complete plan of action for every situation in which the player might be called upon to act, either alone or simultaneously with other players.
Remember! A strategy must specify an action **even** in histories that this same strategy is precluding!

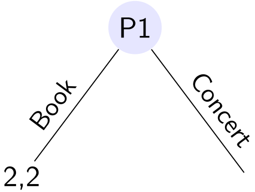
A variant of the Battle of the Sexes



		Player 2	
		B	S
Player 1	B	$3,1$	$0,0$
	S	$0,0$	$1,3$

Strategies?

A variant of the Battle of the Sexes



		Player 2	
		B	S
Player 1	B	$3, 1$	$0, 0$
	S	$0, 0$	$1, 3$

Strategies? Player 1: $(Book, B)$; $(Book, S)$; $(Concert, B)$; $(Concert, S)$
Player 2: B; S

A variant of the Battle of the Sexes

- How do we solve games such as this one?

What strategies do we predict that the players will adopt?
We can start by looking for the Nash equilibria.

- The strategic form of the extensive game

		Player 2: B	Player 2: S
Player 1	$(Concert, B)$	$3, 1$	$0, 0$
	$(Concert, S)$	$0, 0$	$1, 3$
	$(Book, B)$	$2, 2$	$2, 2$
	$(Book, S)$	$2, 2$	$2, 2$

What are the NE?

A variant of the Battle of the Sexes

- How do we solve games such as this one?

What strategies do we predict that the players will adopt?
We can start by looking for the Nash equilibria.

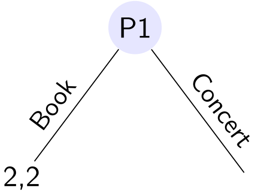
- The strategic form of the extensive game

Player 2

Player 1		B	S
	$(Concert, B)$	$3, 1$	$0, 0$
	$(Concert, S)$	$0, 0$	$1, 3$
	$(Book, B)$	$2, 2$	$2, 2$
	$(Book, S)$	$2, 2$	$2, 2$

What are the NE? $((Concert, B), B)$; $((Book, B), S)$; $((Book, S), S)$

A variant of the Battle of the Sexes



		Player 2	
		B	S
Player 1	B	$3, 1$	$0, 0$
	S	$0, 0$	$1, 3$

The Nash equilibria are:
$((Concert, B), B); ((Book, B), S); ((Book, S), S)$

A variant of the Battle of the Sexes

However, NE might be making "too many" predictions.
Remember: NE does not allow players to modify their strategies as the game unfolds. It assumes a perfect commitment at the beginning of the game.
In particular: the Nash equilibrium $((Book, B), S)$ assumes that the players would not coordinate if they were to play the BoS. This strategy fails to capture the dynamics of the game. (It is not sequentially rational.)

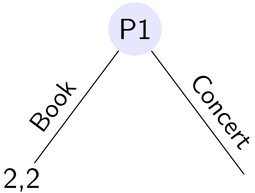
We can try to find the subgame perfect equilibria.

Definition

A subgame perfect equilibrium (SPE) is a profile of strategies that induces a Nash equilibrium (NE) in every subgame.

We can use backwards induction to find the SPE.

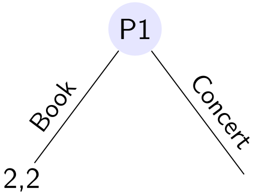
A variant of the Battle of the Sexes



		Player 2: B	Player 2: S
Player 1	B	$3, 1$	$0, 0$
	S	$0, 0$	$1, 3$

What are the SPE?

A variant of the Battle of the Sexes



Player 2

		B	S
Player 1	B	$3,1$	$0,0$
	S	$0,0$	$1,3$

What are the SPE? $((Concert, B), B)$; $((Book, S), S)$

Exercise 211.1: Timing to Claim

- Two players make a joint investment
  - The longer they cooperate with each other, the larger the value of the joint investment
  - However, each player has an incentive to quit cooperating and take all the investment for herself.
  - So at each point in time, players must decide whether to quit or to cooperate.

How long can cooperation be sustained?

Exercise 211.1: Timing to Claim

- The rules

  Players: The set of players is (Player 1, Player 2)

  Sequences: The game has a maximum of T periods (i.e, the largest terminal history is of length T). Players make decisions in each period $t = (2, 3, \ldots T)$ after the initial investment of period 1.

  Player function: $P(h)$=(Player 1, Player 2) for every proper subhistory h.

  Actions: $A_1(h)$=$A_2(h)$=($Quit$, $Not\ Quit$).

  Preferences: The investment starts at \$2 in period 1. In each subsequent period the investment increases by \$2 so that the value of the investment is \$2t in period t. At period t, if player i quits but player j does not, then i takes the whole investment \$2t. If both players quit, then both take half of the investment \$$t$. If both players decide to cooperate, they move on the next period and the investment increases by \$2, unless it is the last period T in which case they both receive \$$T$.

Equilibrium in period T:

		Player 2	
		Q	N
Player 1	Q	T, T	$2T, 0$
	N	$0, 2T$	T, T

Equilibrium in period T:

		Player 2	
		Q	N
Player 1	Q	T, T	$2T, 0$
	N	$0, 2T$	T, T

Equilibrium in period $T-1$:

		Player 2	
		Q	N
Player 1	Q	$T-1, T-1$	$2(T-1), 0$
	N	$0, 2(T-1)$	T, T

Equilibrium in period T:

Player 2

Player 1		Q	N
	Q	T, T	$2T, 0$
	N	$0, 2T$	T, T

Equilibrium in period $T - 1$:

Player 2

Player 1		Q	N
	Q	$T - 1, T - 1$	$2(T - 1), 0$
	N	$0, 2(T - 1)$	T, T

Equilibrium in period 2:

Player 2

Player 1		Q	N
	Q	$2, 2$	$4, 0$
	N	$0, 4$	$3, 3$

Equilibrium in period T:

Player 2

Player 1		Q	N
	Q	T, T	$2T, 0$
	N	$0, 2T$	T, T

Equilibrium in period $T-1$:

Player 2

Player 1		Q	N
	Q	$T-1, T-1$	$2(T-1), 0$
	N	$0, 2(T-1)$	T, T

Equilibrium in period 2:

Player 2

Player 1		Q	N
	Q	$2, 2$	$4, 0$
	N	$0, 4$	$3, 3$

The SPE is $Quit$ in each period for each player.

Electoral competition with strategic voters.

- Players: $n > 2$ candidates and m voters
- Terminal histories: all sequences (x, v) where x is a list of candidates' positions or exit decisions and v is a list of the voters' voting decisions
- Actions:
  - Candidates' possible actions: $x = A_i(\varnothing) = [0, 1] \cup \{Out\}$
  - Voters' possible actions: $v = A_j(x) = \{i \leq n | A_i(\varnothing) \neq \{Out\}\}$
- Preferences:
  - Candidates: n if sole winner, $n - k$ if k candidates tie, 0 if out, and -1 if loses
  - Voters: Each voter has an ideal policy position. She has single-peaked preferences around that ideal policy.

Electoral competition with strategic voters.

- This game has both sequential moves and simultaneous moves.

Electoral competition with strategic voters.

- This game has both sequential moves and simultaneous moves.
  - **Stage 1**: Candidates move simultaneously

Electoral competition with strategic voters.

- This game has both sequential moves and simultaneous moves.
  - **Stage 1**: Candidates move simultaneously
  - **Stage 2**: Candidates' decisions are observed. Then voters move simultaneously.

Electoral competition with strategic voters.

- This game has both sequential moves and simultaneous moves.
  - **Stage 1**: Candidates move simultaneously
  - **Stage 2**: Candidates' decisions are observed. Then voters move simultaneously.
- We must eliminate the weakly dominated strategies of voters to avoid having "too many" equilibria

Electoral competition with strategic voters.

- This game has both sequential moves and simultaneous moves.
  - **Stage 1**: Candidates move simultaneously
  - **Stage 2**: Candidates' decisions are observed. Then voters move simultaneously.
- We must eliminate the weakly dominated strategies of voters to avoid having "too many" equilibria
  - This will usually be the case in games with majority voting, because in most configurations of strategies a single voter will not be pivotal.

Electoral competition with strategic voters.

- This game has both sequential moves and simultaneous moves.
  - **Stage 1**: Candidates move simultaneously
  - **Stage 2**: Candidates' decisions are observed. Then voters move simultaneously.
- We must eliminate the weakly dominated strategies of voters to avoid having "too many" equilibria
  - This will usually be the case in games with majority voting, because in most configurations of strategies a single voter will not be pivotal.
- We can show that in any subgame perfect equilibrium without weakly dominated strategies all the candidates that enter locate at the median.

Committee decision making

- How are bills made into law?

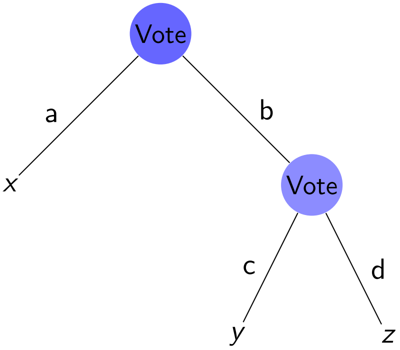
Many legislatures have a system based on committees where a small group of representatives must choose a bill to make into law. Those representatives usually decide by majority voting.

How can we model the decision-making process in a committee?

Committee decision making

Three alternatives: x, y, z
Three committee members: $1, 2, 3$
Assume the committee's procedure is a *binary agenda*
Binary agenda: The first vote is between x and $not\ x$. If a majority votes for x then that alternative wins. If a majority votes for $not\ x$ then a second vote takes place between y and z.

- How can we represent this game graphically?
  - We use a variant of the extensive form game.

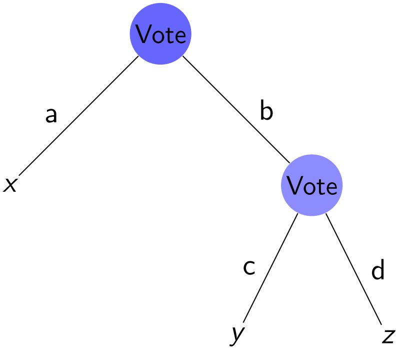


Committee decision making

Assume the following preferences:

Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$

- How can we solve this game?
  - We use backwards induction
  - But we eliminate weakly dominated strategies



Committee decision making

- Look at the last period:

Vote between y and z
We note that y would beat z

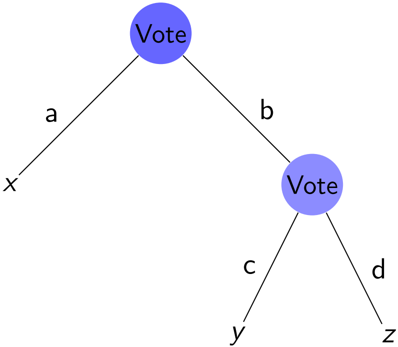
- Look at the first vote:

(This is the reduced game)
Vote between x and y
We note that x would beat y

- The winning alternative:

The SPE is x

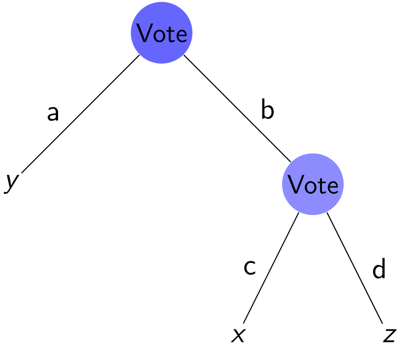
Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$



Committee decision making

- But what if we change the agenda?
  Agenda: the sequence of alternatives to be voted on.

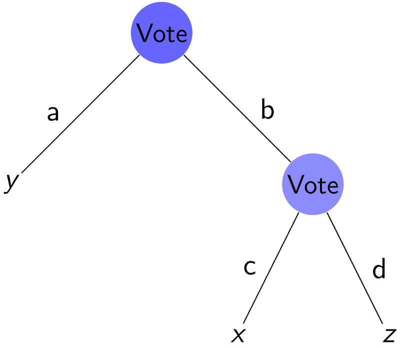
Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$



Committee decision making

- But what if we change the agenda?
  Agenda: the sequence of alternatives to be voted on.
- Consider the sequence:

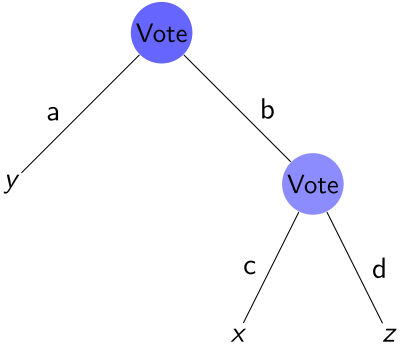
Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$



Committee decision making

- But what if we change the agenda?
  Agenda: the sequence of alternatives to be voted on.
- Consider the sequence:
  - First y against $not\ y$

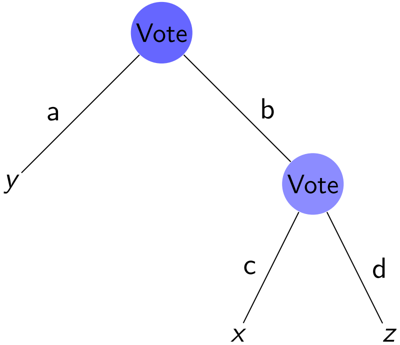
Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$



Committee decision making

- But what if we change the agenda?
  Agenda: the sequence of alternatives to be voted on.
- Consider the sequence:
  - First y against *not* y
  - Then x against z

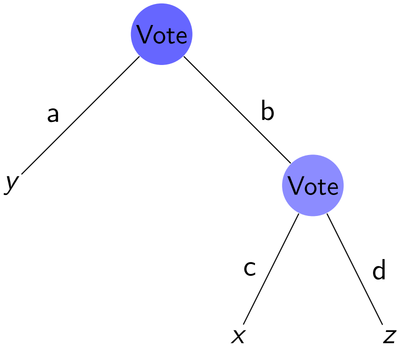
Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$



Committee decision making

- But what if we change the agenda?
  Agenda: the sequence of alternatives to be voted on.
- Consider the sequence:
  - First y against $not\ y$
  - Then x against z
- What is the equilibrium?

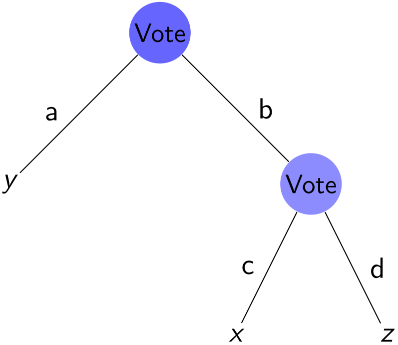
Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$



Committee decision making

- But what if we change the agenda?
  Agenda: the sequence of alternatives to be voted on.
- Consider the sequence:
  - First y against $not\ y$
  - Then x against z
- What is the equilibrium?
  - The winner is y

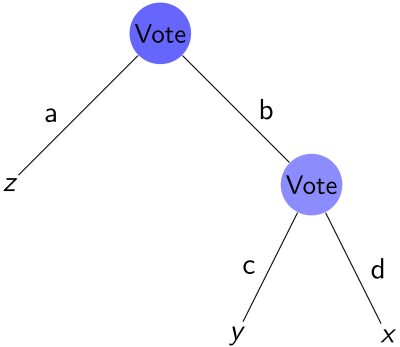
Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$



Committee decision making

Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$

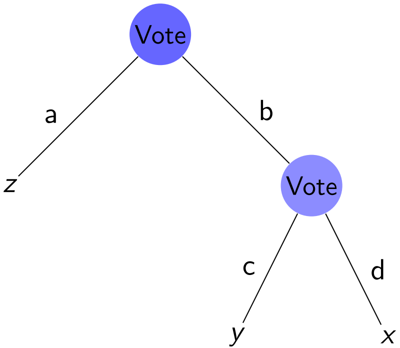
- Now consider the sequence:



Committee decision making

Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$

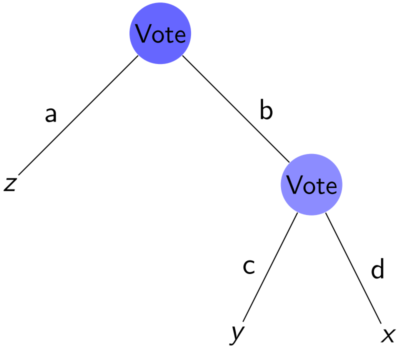
- Now consider the sequence:
  - First z against $not\ z$



Committee decision making

Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$

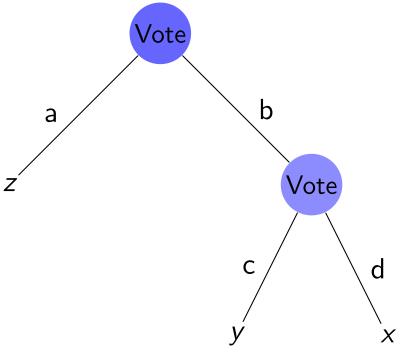
- Now consider the sequence:
  - First z against $not\ z$
  - Then y against x



Committee decision making

Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$

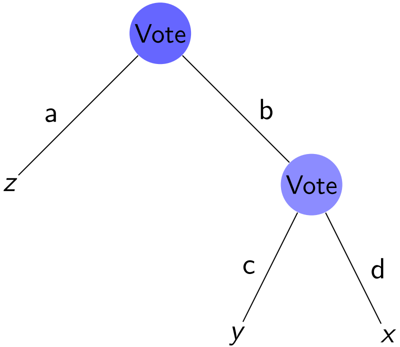
- Now consider the sequence:
  - First z against *not* z
  - Then y against x
- What is the equilibrium?



Committee decision making

Member 1: $x \succ y \succ z$
Member 2: $y \succ z \succ x$
Member 3: $z \succ x \succ y$

- Now consider the sequence:
  - First z against *not* z
  - Then y against x
- What is the equilibrium?
  - The winner is z



Committee decision making

- So the outcome depends on the agenda!

Committee decision making

- So the outcome depends on the agenda!
- What does that say about democracy?

Committee decision making

- So the outcome depends on the agenda!
- What does that say about democracy?
  - Can we trust majority voting?

Committee decision making

- So the outcome depends on the agenda!
- What does that say about democracy?
  - Can we trust majority voting?
  - Does it reflect the "will of the people"?

Committee decision making

- So the outcome depends on the agenda!
- What does that say about democracy?
  - Can we trust majority voting?
  - Does it reflect the "will of the people"?
  - Or does it reflect the will of the agenda-setter?

Committee decision making

- So the outcome depends on the agenda!
- What does that say about democracy?
  - Can we trust majority voting?
  - Does it reflect the "will of the people"?
  - Or does it reflect the will of the agenda-setter?
- This illustrates the large power that comes with setting the agenda of actions to take place.

The Top Cycle Set

The set of alternatives that x beats indirectly, i.e., x beats y if we can construct a chain of alternatives $z_1, z_2, z_3, \ldots z_k$ such that x beats z_1, z_1 beats z_2, ... and z_k beats y.

Exercise 221.1 A committee has three members and there are five alternatives. $P1 : x > y > v > w > z$, $P2 : z > x > v > w > y$, $P3 : y > z > w > v > x$

The Top Cycle Set

The set of alternatives that x beats indirectly, i.e., x beats y if we can construct a chain of alternatives $z_1, z_2, z_3, \ldots z_k$ such that x beats z_1, z_1 beats z_2, ... and z_k beats y.

Exercise 221.1 A committee has three members and there are five alternatives. $P1 : x > y > v > w > z$, $P2 : z > x > v > w > y$, $P3 : y > z > w > v > x$

Find the top cycle set?

The Top Cycle Set

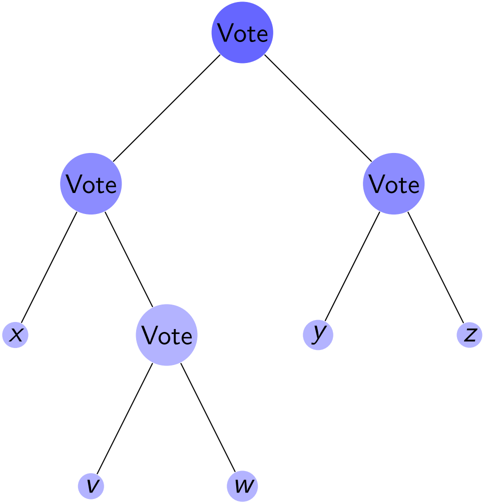
The set of alternatives that x beats indirectly, i.e., x beats y if we can construct a chain of alternatives $z_1, z_2, z_3, \ldots z_k$ such that x beats z_1, z_1 beats z_2, ... and z_k beats y.

Exercise 221.1 A committee has three members and there are five alternatives. $P1 : x > y > v > w > z$, $P2 : z > x > v > w > y$, $P3 : y > z > w > v > x$

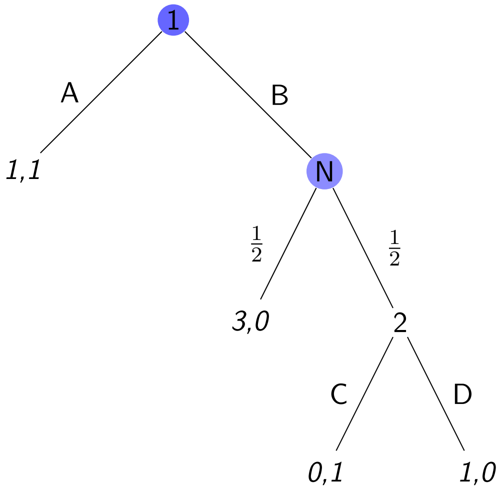
Find the top cycle set?

Design binary agendas that yield each alternative in the top cycle set as an outcome

The Top Cycle Set



Exogenous uncertainty



Exercise 227.3: A Duel

- The Game:
  - Players: The two people.
  - Terminal histories: All sequences of the form $(X_1, X_2, \ldots, X_k, S, H)$, where each X_i is either N ("don't shoot") or (S,M) ("shoot", "miss"), and H means "hit", together with the infinite sequence $(S, M, S, M, S, M, ...)$.
  - Player function: $P(h) = 1$ for any history h that ends in M or N and in which the total number of S's and N's is even, $P(h) = 2$ for any history h that ends in M or N and in which the total number of S's and N's is odd, and $P(h) = c$ for any history h that ends in S.
  - Chance probabilities: Whenever chance moves after amove of player 1 it chooses H with probability p_1 and M with probability $1 - p_1$; whenever it moves after a move of player 2 it chooses H with probability p_2 and M with probability $1 - p_2$;
  - Preferences: Each player's preferences are represented by the expected value of a Bernoulli payoff function that assigns 1 to any history in which she survives and 0 to any history in which she is killed.

Repeated Games: The Prisoner's Dilemma

		Player 2	
		$Quit$	$Fink$
Player 1	$Quit$	$3,3$	$1,6$
	$Fink$	$6,1$	$2,2$

What if Players interact repeatedly?

- You care about the future
- Past actions may affect future behaviour of players

Consider the 'Grim Trigger Strategy':

C as long as the other player(s) cooperate

D forever if a player has defected

- Note – strategy defines actions in any history

Homework Assignment

Question Numbers from Osborne:

- 211.2
- 217.1
- 431.1
- 454.2