Games With Imperfect Information 2: Extensive Form Games, Examples and Applications

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Today

- Clarify indifference condition from last time
- Extend settings of uncertainty
- Extensive form games with uncertainty
- Representation, game trees
- Solution concepts" PBNE, Sequential Equilibria
- Applications

Clarify indifference condition

If agent i is indifferent between two actions, a and b, then the expected utility is the same: $u_i(a,.) = u_i(b,.)$
Example from last time:

- Indifferent (contribute or not) if $E[u_i(1,s_{-i})] = E[u_i(0,s_{-i})]$
- $$E[u_i(1,s_{-i})] = (1-c_i)Pr[\sum_{j\neq i} s_j \geq k-1] + (-c_i)(1-Pr[\sum_{j\neq i} s_j \geq k-1])$$
- $$E[u_i(0,s_{-i})] = (1-c_i)Pr[\sum_{j\neq i} s_j \geq k] + (-c_i)(1-Pr[\sum_{j\neq i} s_j \geq k])$$
- Reduces to: $(1-\hat{c}_n)^{n-1} = \hat{c}_n$ i.e. expected utility not just about probability of being pivotal - also about payoff (i.e. consequence) if your action matters

Extending Last Time

Uncertainty + sequential moves
last time: uncertainty (of payoffs, of state of nature, etc.) in static games (i.e. normal form games)

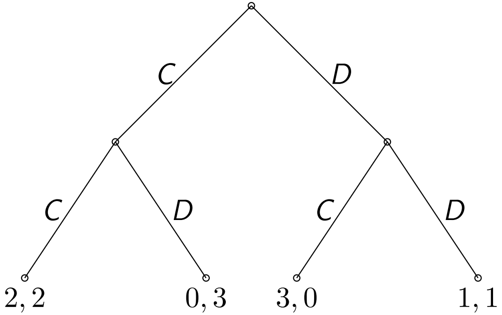
Definition

An extensive form game (in full generality) is:

- Set of agents, N
- Set of histories H (think: nodes in a tree) with (1) initial history $H^0 = \emptyset$ and (2) terminal histories H^T
- Order of moves (think: who moves when): $p(h) : H \backslash H^T \rightarrow N$
- Actions available to each player: $A(h)$ that player at history h (this player is given by p(h)) may take
- Information sets $I \subset H \backslash H^T$: partition H (think: if player i = p(h)) is called to move at information set I_h, she doesn't know which node in I_h she is at
- Payoffs: $u_i : H^T \rightarrow \mathbb{R}$
- (See M&M pg. 173 or Os pg. 314 for formal requirements for tree i.e. no cyles, two nodes don't lead to same node, etc.)

Illustration 1: Prisoner's Dilemma

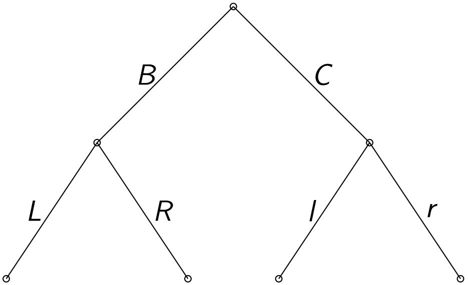
	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$



Strategies

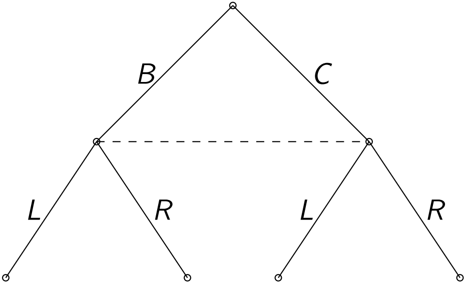
Let H_i be the set of histories for which player i is called to act:
$H_i = \{h|p(h) = i\}$
A strategy (for player i) in an extensive form game is a map $s_i : H_i \rightarrow A(h)$ such that $s_i(h) = s_i(h')$ if h and h' are in the same info set.

Illustration 2: strategies



Example strategy: $s_1 = (B), s_2 = (L, r)$

Illustration 2: strategies



Example strategy: $s_1 = (B), s_2 = (R)$

Solution Concepts

Idea: Nash Eq requires two things. (1) best response to beliefs and (2) correct beliefs, in eq. In extensive form games: Subgame perfection requires *credible* actions and beliefs. We can extend these ideas naturally to games in general:

- Bayes Nash Eq (last time): Nash eq with utility replaced by expected utility
- Perfect Bayes Nash Eq (PBNE): Requires actions and beliefs to be consistent at all subgames (subgame starts at a node, not an info set)
- Sequential Equilibria: require actions and belief to be a stronger notion of consistent (PBNE only applies to info sets reached with positive probability, might want something stronger)

Solution Concept 1: PBNE

A PBNE is two things:

- Strategy: for each info set, a (probabilistic) prescription of what to do next and
- Beliefs: probability distribution over each info set. i.e. for each node I could be at, what probability that I am actually at that node?

such that

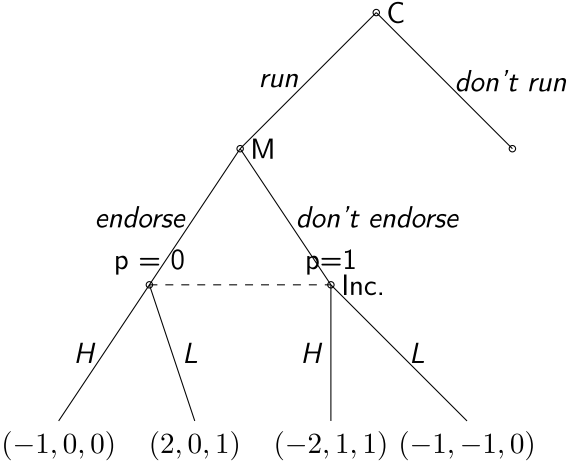
- Strategies are rational - optimal given beliefs
- Beliefs are consistent with the strategy profile (i.e. obtained via Bayes rule - when applicable - and strategy profile)

Illustration: PBNE

Players: Challenger (C), Media (M) and Incumbent (Inc).

- Challenger runs or not
- Media endorses or not and Incumbent puts in High Effort or Low Effort
- Preferences;
  - C wants to enter if (i) media endorses and (ii) Inc. low effort;
  - Media wants to endorse C only if Inc is low effort;
  - Inc. wants to exert effort only if C not endorsed

Illustration: PBNE, Cont.



Consider strategy: ($s_C =$ don't run; $s_M =$ endorse if C runs and; $s_{Inc} = H$ if M endorses) and beliefs Pr[Media endorses Challenger] = 0. This is a PBNE. This is not subgame perfect. Why?

Sol.Con.2: Sequential Equilibria

Definition:
For a finite extensive form game with imperfect information, a sequential equilibria is a pair (β, μ) such that (1) β is sequentially rational and (2) there exist a sequence of fully mixed strategies and beliefs converging to β, μ that are PBNE[1]

[1] eventually - see def. 8.6 in M&M (pg. 238) if you re really interested in the technical condition eventually.

Example: Strategic Information Transmission

Set up: Legislature (L) and bureaucracy (B) with different preferences.
B observes some information, $t \in [0,1]$ (think (true) good policy), and communicates some message r to L.
L has prior beliefs $t \sim U[0,1]$ and takes some action y (sets policy, enforcement level, etc.)
Payoffs:

$$u_B(y,t) = -(y-(t-b))^2 \tag{1}$$

$$u_L(y,t) = -(y-t)^2 \tag{2}$$

Picture

Example, Cont.

Look for PBNE.

- Full information transmission: can we have $r(t) = t$?
  - Well, if we did, L implements $y = t$. If so, is B best responding? No!
- No information transmission: can we have $r(t) = c$?
  - If so, L gets no info (still believes $t \sim U[0,1]$) and BR with

$$y = 0.5(= argmax_y \int_0^1 -(y-t)^2.1dt) \tag{3}$$

- (what does L believe after receiving $c' \neq c$? Does it matter? Assume L beliefs are constant.
- If L beliefs are constant, B cannot influence L action $\rightarrow r(t) = c$ a best response

Example, Cont.: Partial information transmission

Look for Eq in which $r(t) = \begin{cases} r_1 & \text{if } 0 \leq t < t_1 \\ r_2 & \text{if } t_1 \leq t < 1 \end{cases}$

- If L receives r_1, what do they believe? $y \sim U[0, t_1]$
- $\rightarrow y^* = t_1/2$
- Similarly, if L receives r_2, takes action $y^* = \frac{1+t_1}{2}$
- Are the L's beliefs constrained after receiving $r \notin \{r_1, r_2\}$? No. So we may consider L having beliefs leading to actions $t_1/2$ or $\frac{1+t_1}{2}$
- For the sender (B): actions (r) can only 'induce' $y = t_1$ or $\frac{1+t_1}{2}$. So, when will B send r_1 or r_2?
- r_1 optimal when $u_B(t_1/2, t) \geq u_B(\frac{1+t_1}{2}, t)$

Example, Cont.: Partial information transmission

- Note: B is indifferent when $u_B(t_1/2, t) = u_B(\frac{1+t_1}{2}, t)$
- $\rightarrow u_B(t_1/2, t) = u_B(\frac{1+t_1}{2}, t)$
- $\rightarrow$ in state t_1, the bureaucrat is indifferent $\rightarrow t_1 + b$ is midway between $t_1/2$ and $\frac{1+t_1}{2}$
- $t_1 = \frac{1}{2} - 2b$
- Note: for the conjectured Eq to exist, we need $t_1 > 0$ so that $b < \frac{1}{4}$

Homework Questions from Osborne

- Exercises 318.2
- Exercise 319.3