Formal Analysis: Lecture 3

Raymond Duch

Nuffield College

February 1, 2010

Illustrations: Hotelling's Model of Electoral Competition

- Two Candidate Election
  - Candidates x_1 and x_2
  - Suppose $x_2 < m$
  - If $x_1 < x_2$ then x_2 attracts all votes $> \frac{1}{2}(x_1 + x_2)$ thus winning>50 percent of votes

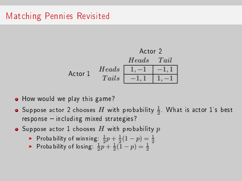
Illustrations: Hotelling's Model of Electoral Competition

- Best response function for Player 1 ($B_1(x_2)$)
  - $x_1 : x_2 < x_1 < 2m - x_2$ if $x_2 < m$
  - m if $x_2 = m$
  - $x_1 : 2m - x_2 < x_1 < x_2$ if $x_2 > m$
- Best response function for Player 2 ($B_2(x_1)$)
  - $x_2 : x_1 < x_2 < 2m - x_1$ if $x_1 < m$
  - m if $x_1 = m$
  - $x_2 : 2m - x_1 < x_2 < x_1$ if $x_1 > m$
- Unique Nash Equilibrium is where both candidates choose the position m

Matching Pennies Revisited

		Actor 2	
		$Heads$	$Tail$
Actor 1	$Heads$	$1, -1$	$-1, 1$
	$Tails$	$-1, 1$	$1, -1$

- How would we play this game?
- Suppose actor 2 chooses H with probability $\frac{1}{2}$. What is actor 1's best response – including mixed strategies?
- Suppose actor 1 chooses H with probability p
  - Probability of winning: $\frac{1}{2}p + \frac{1}{2}(1 - p) = \frac{1}{2}$
  - Probability of losing: $\frac{1}{2}p + \frac{1}{2}(1 - p) = \frac{1}{2}$



Matching Pennies Revisited

		Actor 2	
		Heads	*Tail*
Actor 1	*Heads*	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

- How would we play this game?
- Suppose actor 2 chooses H with probability $\frac{1}{2}$. What is actor 1's best response – including mixed strategies?
- Suppose actor 1 chooses H with probability p
  - Probability of winning: $\frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2}$
  - Probability of losing: $\frac{1}{2}p + \frac{1}{2}(1-p) = \frac{1}{2}$

Player 2? Equilibrium when both mix with probability $\frac{1}{2}$.

Matching Pennies Revisited

		Actor 2	
		$Heads$	$Tail$
Actor 1	$Heads$	$1, -1$	$-1, 1$
	$Tails$	$-1, 1$	$1, -1$

- Other equilibria?
  - Suppose player 2 mixes with probability q
    - The probability that 1 wins equals: $pq + (1-p)(1-q)$
    - & loses with probability: $p(1-q) + (1-p)q$
    - simplifies to $1 - q + p(2q-1)$ and $q + p(1-2q)$
  - If $q < \frac{1}{2}$ then Player 1 chooses Tails

Notes re: Mixed Strategy Equilibrium

Now if $q < \frac{1}{2}$ then $2q - 1$ is negative and choosing a higher p will reduce the likelihood of winning. Thus, player 1 would choose $p = 0$. Now suppose $q > \frac{1}{2}$ – then $2q - 1$ is positive and choosing a higher p increases the probability of winning, i.e, player 1 will choose $p = 1$. Thus, there are no other mixed strategy equilibria.

Mixed Strategies: Definitions

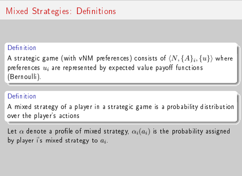
Definition

A strategic game (with vNM preferences) consists of $\langle N, \{A\}_i, \{u\} \rangle$ where preferences u_i are represented by expected value payoff functions (Bernoulli).

Definition

A mixed strategy of a player in a strategic game is a probability distribution over the player's actions

Let α denote a profile of mixed strategy, $\alpha_i(a_i)$ is the probability assigned by player i's mixed strategy to a_i.



Mixed Strategies: Definitions

Definition

A strategic game (with vNM preferences) consists of $\langle N, \{A\}_i, \{u\} \rangle$ where preferences u_i are represented by expected value payoff functions (Bernoulli).

Definition

A mixed strategy of a player in a strategic game is a probability distribution over the player's actions

Let α denote a profile of mixed strategy, $\alpha_i(a_i)$ is the probability assigned by player i's mixed strategy to a_i.

Shorthand – in same order as actions are listed in game, i.e.$(.25, .75)$.

Must sum to one – mixed strategies include pure strategies.

Nash Equilibrium in Mixed Strategies

Equilibrium and best responses are defined in the same manner as before.

Definition (Nash Equilibrium in Mixed Strategies)

$u_i(\alpha^*) \geq u_i(\alpha_i, \alpha^*_{-i})$, $\forall \alpha_i \in A_i$ and $\forall i \in N$

The mixed strategy profile α^* is a mixed strategy profile if and only if α^* is in $B_i(\alpha^*_{-i}), \forall i \in N$.

Nash Equilibrium in Mixed Strategies

Consider a two player strategic game. Assume P1 plays T with probability p and P2 plays L with q.

		Actor 2	
		L	R
Actor 1	T	pq	$p(1-q)$
	B	$(1-p)q$	$(1-p)(1-q)$

P 1's expected payoff is then:

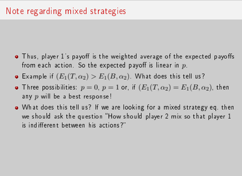
$$pqu_1(T,L)+p(1-q)u_1(T,R)+(1-p)qu_1(B,L)+(1-p)(1-q)u_1(B,R)$$

or $p[qu_1(T,L)+(1-q)u_1(T,R)]+(1-p)[qu_1(B,L)+(1-q)u_1(B,R)]$

or $p[E_1(T,\alpha_2)]+(1-p)[E_1(B,\alpha_2)]$

Note regarding mixed strategies

- Thus, player 1's payoff is the weighted average of the expected payoffs from each action. So the expected payoff is linear in p.
- Example if $(E_1(T, \alpha_2) > E_1(B, \alpha_2)$. What does this tell us?
- Three possibilities: $p = 0$, $p = 1$ or, if $(E_1(T, \alpha_2) = E_1(B, \alpha_2)$, then any p will be a best response!
- What does this tell us? If we are looking for a mixed strategy eq. then we should ask the question “How should player 2 mix so that player 1 is indifferent between his actions?”



Note regarding mixed strategies

- Thus, player 1's payoff is the weighted average of the expected payoffs from each action. So the expected payoff is linear in p.
- Example if $(E_1(T,\alpha_2) > E_1(B,\alpha_2)$. What does this tell us?
- Three possibilities: $p=0$, $p=1$ or, if $(E_1(T,\alpha_2) = E_1(B,\alpha_2)$, then any p will be a best response!
- What does this tell us? If we are looking for a mixed strategy eq. then we should ask the question “How should player 2 mix so that player 1 is indifferent between his actions?”

Thus, player 1's payoff is the weighted average of the expected payoffs from each action. So the expected payoff is linear in p.
Example if $(E_1(T,\alpha_2) > E_1(B,\alpha_2)$. What does this tell us? Three possibilities: $p=0$, $p=1$ or, if $(E_1(T,\alpha_2) = E_1(B,\alpha_2)$, then any p will be a best response!
What does this tell us? If we are looking for a mixed strategy eq. then we should ask the question “How should player 2 mix so that player 1 is indifferent between his actions?”

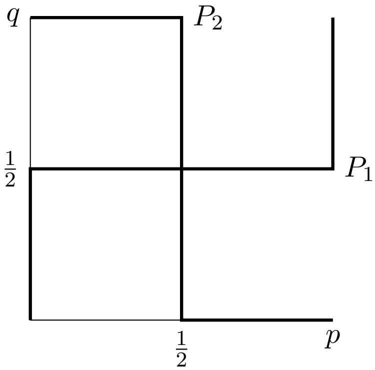
Back to Matching Pennies

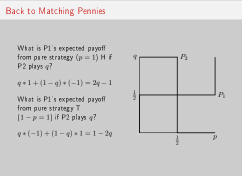
What is P1's expected payoff from pure strategy ($p = 1$) H if P2 plays q?

$$q * 1 + (1 - q) * (-1) = 2q - 1$$

What is P1's expected payoff from pure strategy T ($1 - p = 1$) if P2 plays q?

$$q * (-1) + (1 - q) * 1 = 1 - 2q$$





Back to Matching Pennies

What is P1's expected payoff from pure strategy $(p = 1)$ H if P2 plays q?

$q * 1 + (1 - q) * (-1) = 2q - 1$

What is P1's expected payoff from pure strategy T $(1 - p = 1)$ if P2 plays q?

$q * (-1) + (1 - q) * 1 = 1 - 2q$

Draw best responses (What if $q = 0$?). When is P1 indifferent? When $2q - 1 = 1 - 2q \rightarrow 4q = 2 \rightarrow q = \frac{1}{2}$.

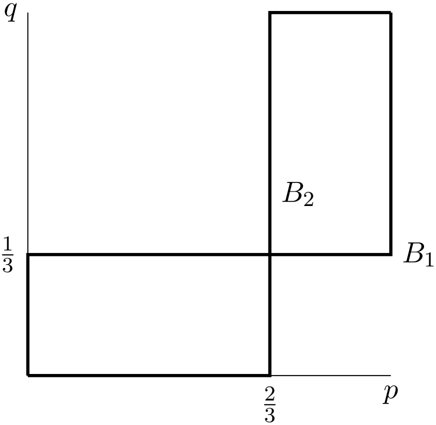
Best Responses for Matching Pennies Game

- Best response function for Player 1
    - $B_1(q) = 0$ if $q < \frac{1}{2}$
    - $B_1(q) = p : 0 \leq p \leq 1$ if $q = \frac{1}{2}$
    - $B_1(q) = 1$ if $q > \frac{1}{2}$
- Best response function for Player 2
    - $B_2(p) = 1$ if $p < \frac{1}{2}$
    - $B_2(p) = p : 0 \leq p \leq 1$ if $p = \frac{1}{2}$
    - $B_2(p) = 0$ if $p > \frac{1}{2}$

Battle of the Sexes revisited

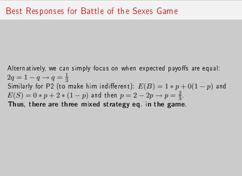
		Actor 2	
		B	S
Actor 1	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

P1 expected payoff to B is $2 * q + 0 * (1 - q) = 2q$ and $0 * q + 1 * (1 - q) = 1 - q$ to S. Thus, P1 prefers B if $2q > 1 - q$ or $q > \frac{1}{3}$ and S if $q < \frac{1}{3}$



Best Responses for Battle of the Sexes Game

Alternatively, we can simply focus on when expected payoffs are equal: $2q = 1 - q \rightarrow q = \frac{1}{3}$
Similarly for P2 (to make him indifferent): $E(B) = 1 * p + 0(1 - p)$ and $E(S) = 0 * p + 2 * (1 - p)$ and then $p = 2 - 2p \rightarrow p = \frac{2}{3}$.
Thus, there are three mixed strategy eq. in the game.



Best Responses for Battle of the Sexes Game

Alternatively, we can simply focus on when expected payoffs are equal:
$2q = 1 - q \rightarrow q = \frac{1}{3}$
Similarly for P2 (to make him indifferent): $E(B) = 1 * p + 0(1 - p)$ and $E(S) = 0 * p + 2 * (1 - p)$ and then $p = 2 - 2p \rightarrow p = \frac{2}{3}$.
Thus, there are three mixed strategy eq. in the game.

Alternatively, we can simply focus on when expected payoffs are equal:
$2q = 1 - q \rightarrow q = \frac{1}{3}$
Similarly for P2 (to make him indifferent): $E(B) = 1 * p + 0(1 - p)$ and $E(S) = 0 * p + 2 * (1 - p)$ and then $p = 2 - 2p \rightarrow p = \frac{2}{3}$.

Best Responses for Battle of the Sexes Game

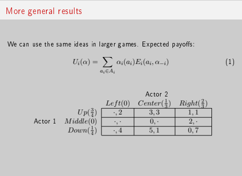
- Best response function for Player 1
  - $B_1(q) = 0$ if $q < \frac{1}{3}$
  - $B_1(q) = p : 0 \leq p \leq 1$ if $q = \frac{1}{3}$
  - $B_1(q) = 1$ if $q > \frac{1}{3}$
- Best response function for Player 2
  - $B_2(p) = 0$ if $p < \frac{2}{3}$
  - $B_2(p) = p : 0 \leq p \leq 1$ if $p = \frac{2}{3}$
  - $B_2(p) = 1$ if $p > \frac{2}{3}$

More general results

We can use the same ideas in larger games. Expected payoffs:

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) E_i(a_i, \alpha_{-i}) \tag{1}$$

		Actor 2		
		$Left(0)$	$Center(\frac{1}{3})$	$Right(\frac{2}{3})$
Actor 1	$Up(\frac{3}{4})$	$\cdot, 2$	$3, 3$	$1, 1$
	$Middle(0)$	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
	$Down(\frac{1}{4})$	$\cdot, 4$	$5, 1$	$0, 7$



More general results

We can use the same ideas in larger games. Expected payoffs:

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) E_i(a_i, \alpha_{-i}) \tag{1}$$

		Actor 2		
		$Left(0)$	$Center(\frac{1}{3})$	$Right(\frac{2}{3})$
Actor 1	$Up(\frac{3}{4})$	$\cdot, 2$	$3, 3$	$1, 1$
	$Middle(0)$	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
	$Down(\frac{1}{4})$	$\cdot, 4$	$5, 1$	$0, 7$

The logic we went through before extends to all strategic form games – in a mixed strategy equilibrium a player will only assign a positive probability to actions that have equal (and highest) expected utility. The payoff to any action that is assigned zero probability can at most be equal to actions that are taken with a positive probability.

Helps us check whether we have found an equilibrium:

Mixed Strategies

Proposition

A mixed strategy profile α^* in a strategic game with vNM preferences in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player i,

- *the expected payoff, given α^*_{-i}, to every action to which α^*_i assigns positive probability is the same*
- *the expected payoff, given α^*_i, to every action to which α^*_i assigns zero probability is at most the expected payoff to any action to which α^*_i assigns positive probability.*

Check: Is mixed strategy profile a mixed strategy Nash Equilibrium?

- For Player 1
  - $U = \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$
  - $M = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$
  - $D = \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 0 = \frac{5}{3}$
- For Player 2
  - $L = \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 4 = \frac{5}{2}$
  - $C = \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 1 = \frac{5}{2}$
  - $R = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 7 = \frac{5}{2}$

Mixed Strategies

Proposition

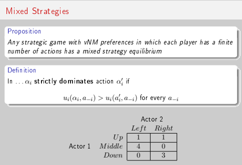
Any strategic game with vNM preferences in which each player has a finite number of actions has a mixed strategy equilibrium

Definition

In . . . α_i **strictly dominates** action α'_i if

$$u_i(\alpha_i, a_{-i}) > u_i(a'_i, a_{-i}) \text{ for every } a_{-i}$$

		Actor 2	
		$Left$	$Right$
Actor 1	Up	1	1
	$Middle$	4	0
	$Down$	0	3



Mixed Strategies

Proposition

Any strategic game with vNM preferences in which each player has a finite number of actions has a mixed strategy equilibrium

Definition

In . . . α_i **strictly dominates** action α_i' if

$$u_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for every } a_{-i}$$

		Actor 2	
		$Left$	$Right$
	Up	1	1
Actor 1	$Middle$	4	0
	$Down$	0	3

Here mixing $\frac{1}{2}$ over $Middle$ and $Down$ dominates Up.
Strictly dominated strategies are never used with a positive probability in any equilibrium.

Mixed Strategies

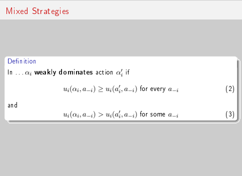
Definition

In . . . α_i **weakly dominates** action α'_i if

$$u_i(\alpha_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \text{ for every } a_{-i} \tag{2}$$

and

$$u_i(\alpha_i, a_{-i}) > u_i(a'_i, a_{-i}) \text{ for some } a_{-i} \tag{3}$$



Mixed Strategies

Definition

In . . . α_i **weakly dominates** action α_i' if

$$u_i(\alpha_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every } a_{-i} \tag{2}$$

and

$$u_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some } a_{-i} \tag{3}$$

We cannot eliminate weakly dominated strategies but there always exists a mixed strategy equilibrium in which no-one uses weakly dominated actions.

An example

		Consumer	
		$Accept(q)$	$Reject(1-q)$
Expert	$Honest(p)$	$\pi, -rE-(1-r)I$	$(1-r)\pi, -rE'-(1-r)I$
	$Dishonest(1-p)$	$r\pi+(1-r)\pi', -E$	$0, -rE'-(1-r)I'$

Expert is indifferent if

$$q\pi + (1-q)(1-r)\pi = q(r\pi + (1-r)\pi')$$

or if

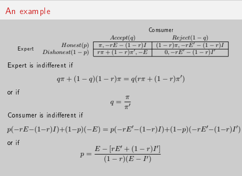
$$q = \frac{\pi}{\pi'}$$

Consumer is indifferent if

$$p(-rE-(1-r)I)+(1-p)(-E) = p(-rE'-(1-r)I)+(1-p)(-rE'-(1-r)I')$$

or if

$$p = \frac{E-[rE'+(1-r)I']}{(1-r)(E-I')}$$



An example

		Consumer	
		$Accept(q)$	$Reject(1-q)$
Expert	$Honest(p)$	$\pi, -rE-(1-r)I$	$(1-r)\pi, -rE'-(1-r)I$
	$Dishonest(1-p)$	$r\pi+(1-r)\pi', -E$	$0, -rE'-(1-r)I'$

Expert is indifferent if

$$q\pi + (1-q)(1-r)\pi = q(r\pi + (1-r)\pi')$$

or if

$$q = \frac{\pi}{\pi'}$$

Consumer is indifferent if

$$p(-rE-(1-r)I)+(1-p)(-E) = p(-rE'-(1-r)I)+(1-p)(-rE'-(1-r)I')$$

or if

$$p = \frac{E-[rE'+(1-r)I']}{(1-r)(E-I')}$$

Problem can be major or minor, probability of a major problem is r. Mechanic can tell and can claim either. If told minor the consumer will always accept. If problem is major then the mechanic will always claim major.

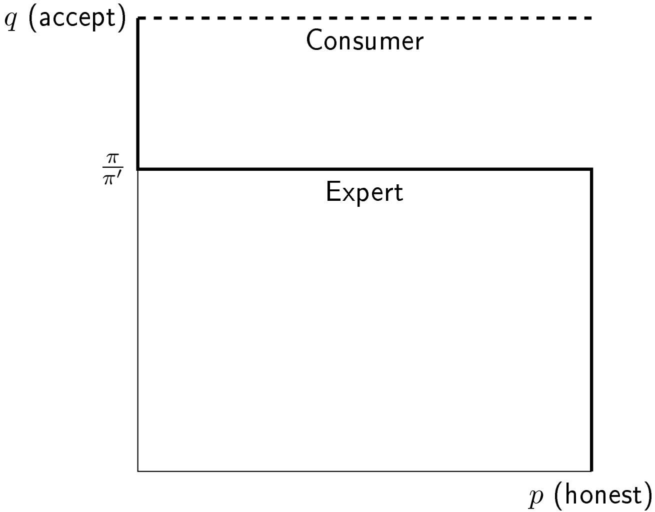
Expert gets π for charging correctly, π' for fixing a minor problem at a major price. The consumer pays E for a major problem and I for a minor problem. If she goes elsewhere she'll end up paying E' and I'.

We can model this game. The expert can play Honest or Dishonest. Consumer chooses Accept or Reject (if major).

If expert is honest, consumer prefers to accept.

If consumer always accepts, then expert benefits from being dishonest.

An example: $E < rE' + (1-r)I'$



An example: $E > rE' + (1-r)I'$