Formal Analysis: Lecture 2

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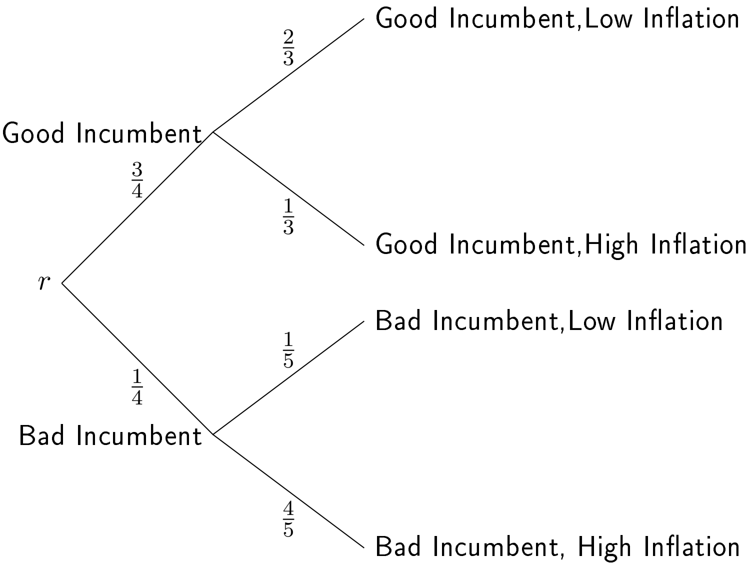
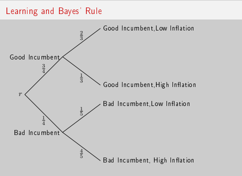
Learning and Bayes' Rule





1. Learning the Incumbents Type.

Bayesian Reasoning

The likelihood the incumbent is good if we observe low inflation:

1. Agent knows that there is a $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$ probability of reaching the top node.
2. And a $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$ probability of reaching the third node.
3. After observing low inflation its 10 times as likely that the incumbent is good.
4. Let $p(l)$ be probability of good incumbent conditional on low inflation.
5. Because probabilities must sum to 1, $p(l) + \frac{p(l)}{10} = 1$ so that $p(l) = \frac{10}{11}$
6. $10p(l) + p(l) = 10$
7. $p(l)(10 + 1) = 10$
8. $p(l) = \frac{10}{11}$

Bayes' Rule

Let $A_1 ... A_N$ be disjoint events (i.e., no two can occur simultaneously) such that $\sum Pr(A_n) = 1$ and $Pr(A_n) > 0$ for all n. Let B be some other event. Then:

$$Pr(A_j|B) = \frac{Pr(B|A_j)Pr(A_j)}{\sum_{n=1}^{N} Pr(B|A_n)Pr(A_n)} \quad (1)$$

Bayes Incumbent/Inflation Example

Returning to our example, let A_1 be the event that the incumbent is good and A_2 be the event that she is bad. Event B is low inflation. The Bayes formulae are:

$$Pr(A_1|B) = \frac{Pr(B|A_1)Pr(A_1)}{Pr(B|A_l)Pr(A_l) + Pr(B|A_2)Pr(A_2)} \quad (2)$$

$$Pr(A_2|B) = \frac{Pr(B|A_2)Pr(A_2)}{Pr(B|A_1)Pr(A_1) + Pr(B|A_2)Pr(A_2)} \quad (3)$$

Bayes Incumbent/Inflation Example

- $Pr(A_1) = \frac{3}{4}$
- $Pr(A_2) = \frac{1}{4}$
- $Pr(B|A_1) = \frac{2}{3}$
- $Pr(B|A_2) = \frac{1}{5}$

$$Pr(A_1|B) = \frac{\frac{2}{3} \times \frac{3}{4}}{\frac{2}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{1}{4}} \tag{4}$$

and

$$Pr(A_2|B) = \frac{\frac{1}{5} \times \frac{1}{4}}{\frac{2}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{1}{4}} \tag{5}$$

Strategic/Normal form games

- A strategic game is defined as:
  - a set of players
  - a set of actions
  - and for each player, preferences over action profiles
- Alternatively, $\langle N, \{A_i, u_i(\cdot \ldots \cdot)\}_i \rangle$
  - N is set of actor $\{1, \ldots, n\}$
  - A_i is the set of actions available to actor i
  - $u_i(\cdot \ldots \cdot)$ is actor i's utility function (preferences), which (normally) depends on the actions of all the actors.

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For short we sometimes write $\langle N, A, u \rangle$.

Strategic/Normal form games

- $a = (a_1, \ldots, a_n)$ denotes a vector of the actors' actions
- a_{-i} (or $a_{\sim i}$) denotes the actions of everyone except actor i
- then $a = (a_i, a_{-i})$

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If there are three players then (a_2, a_{-2}) is the action profile where player 1 and 3 stick with the original action profile but player 2 deviates to some action.

Nash Equilibrium

Consider a familiar game:

		Suspect 2	
		Quiet	*Fink*
Suspect 1	*Quiet*	$-4, -4$	$-25, -1$
	Fink	$-1, -25$	$-8, -8$

Here $N = \{1, 2\}, A_1 = A_2 = \{\text{Quiet}, \text{Fink}\}$ and, e.g., u_1(Fink, Quiet)=-1

Definition (Nash Equilibrium)

$$\forall i \in N, u_i(a^*) \geq u_i(a_i, a^*_{-i}), \forall a_i \in A_i \quad (6)$$

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Given that the actors know the structure of the game and forms beliefs about how the other actor will act we can make predictions about the equilibrium. A Nash equilibrium equilibrium is an action profile that no actor can do better by unilaterally changing his action.

Dominated actions

Are there any actions such that we would always prefer taking some other action instead? Formally: action a_i'' dominates action a_i' if

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \quad (7)$$

Consider this game:

		Actor 2		
		$Left$	$Center$	$Right$
	Up	$2,1$	$0,0$	$2,1$
Actor 1	$Middle$	$2,4$	$0,0$	$6,3$
	$Down$	$1,0$	$0,0$	$5,1$

Battle of the sexes/Bach or Stravinsky

Actor 2

Actor 1		*Bach*	*Stravinsky*
	Bach	$2, 1$	$0, 0$
	Stravinsky	$0, 0$	$1, 2$

Matching Pennies

		Actor 2	
		$Heads$	$Tail$
Actor 1	$Heads$	$1, -1$	$-1, 1$
	$Tails$	$-1, 1$	$1, -1$

HI-LO: Coordination

		Actor 2	
		$Stag$	$Hare$
Actor 1	$Stag$	$2, 2$	$0, 1$
	$Hare$	$1, 0$	$1, 1$

Best Response

Definition (Best Responses)

A players best response function (correspondence) specifies the the best response to given set of actions by the other players:

$$B_i(a_{-i}) = \{a_i \in A_i | u_i(a_i, a_{-i}) \geq u_(a'_i, a_{-i}) \ \forall \ a'_i \in A_i\}$$

The best response is set valued (a correspondence).

Definition (Best Response Def of NE)

The action profile a^* is a NE iff

$$a^*_i \in B_i(a^*_{-i}), \ \forall i \in N$$

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An easier way to find equilibria in more complicated games.

Best Response

		Actor 2		
		Left	*Center*	*Right*
Actor 1	*Up*	$1, 2^*$	$2^*, 1$	$1^*, 0$
	Middle	$2^*, 1^*$	$0, 1^*$	$0, 0$
	Down	$0, 1$	$0, 0$	$1^*, 2^*$

- What does actor 1 do if actor 2 is playing “Left”?
- What does actor 1 do if actor 2 is playing “Center”?
- etc. . .

Best Response and Nash Equilibria

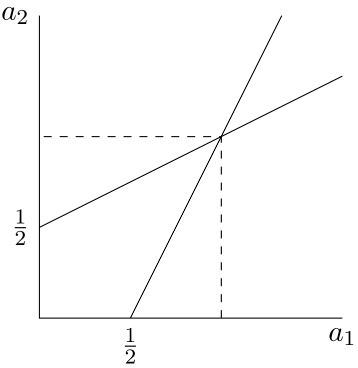
- $a_1^* = b_1(a_2^*)$
  - (M,L)
  - (T,C)
  - (T,R)
  - (T,B)
- $a_2^* = b_2(a_1^*)$
  - (T,L)
  - (M,L)
  - (M,C)
  - (B,R)
- (a_1^*, a_2^*)
  - (M,L)
  - (B,R)

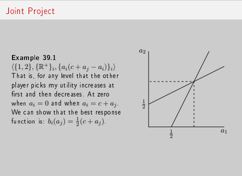
Joint Project

Example 39.1

$\langle \{1, 2\}, \{\mathbb{R}^+\}_i, \{a_i(c + a_j - a_i)\}_i \rangle$

That is, for any level that the other player picks my utility increases at first and then decreases. At zero when $a_i = 0$ and when $a_i = c + a_j$. We can show that the best response function is: $b_i(a_j) = \frac{1}{2}(c + a_j)$.





The intersection of the lines represents the equilibrium – where the players' actions are mutual best responses. We can solve this analytically. The best response functions are:

$$a_1 = \frac{1}{2}(c + a_2) \tag{8}$$

$$a_2 = \frac{1}{2}(c + a_1) \tag{9}$$

Solve for $a_1, a_2 \rightarrow a_1 = a_2 = c$

Solving for Nash Equilibria

- $a_1 = \frac{1}{2}(c + a_2)$
- $a_2 = \frac{1}{2}(c + a_1)$
- substituting we get $a_1 = \frac{1}{2}(c + \frac{1}{2}(c + a_1))$
- $a_1 = \frac{3}{4}c + \frac{1}{4}a_1$
- so that $a_1 = c$

Strictly Dominated Strategies

Strick domination

A strictly dominated action is not a best response to any actions of the other players: whatever the other players do, some other action is better. Are there any actions such that we would always prefer taking some other action instead? Formally: action a_i'' strictly dominates action a_i' if

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{10}$$

We say that a' is **strictly dominated**. A strictly dominated action is not used in any NE.

Strictly Dominated Strategies

Weak domination
Formally: In a strategic game with ordinal preferences, player i's action a''_i **weakly dominates** her action a'_i if

$$u_i(a''_i, a_{-i}) \geq u_i(a'_i, a_{-i}), \forall a_{-i} \in A_{-i} \quad (11)$$

for every list a_i of the players' actions
and

$$u_i(a''_i, a_{-i}) > u_i(a'_i, a_{-i}), \forall a_{-i} \in A_{-i} \quad (12)$$

for some list a_{-i} of the other players' actions.
We say that a' is **weakly dominated**.

Guessing Game: CESS Experiment

Guessing game:

- All participants in one session took part in the same guessing game.
- They had to guess the closest number to two-thirds of the average guess.
- The winner won £20, the rest nothing.
- In the event of ties, one of the winners was selected randomly.

Guessing Game: Nash Equilibrium

- the unique equilibrium for the Beauty-contest is 0
- typically the percentage of subjects choosing 0 in experiments is less than 10 percent
- only one non-student subject in our experiment selected 0
- The average guess of our student subjects is 38 which is consistent with similar guessing games results
- The average guess of our non-student subject pool was 46