Formal Analysis: Lecture 1

Raymond Duch

Nuffield College

January 18, 2010

Topics

- Notation
- How to write mathematics
- Rationality
- Utility
- Uncertainty
- Risk Preferences
- Methods for aggregating collective preferences

Logic Symbols

- $\in$ In
- $\forall$ For all
- $\exists$ There exists....
- $\exists!$ There exists a unique...
- $\neg$ Not
- $\sim$ Not
- ! Not
- $\vee$ Or
- $\wedge$ And
- $|$ Such that, given that
- : Such that, given that
- $\times$ The Cartesian product / Cross product: e.g., the set A $\times$ is the set of all pairs (a,b) in which a $\in$ A and b $\in$ B

How to write mathematics

- Effective mathematical communications
- Word versus symbols
- Equalities
- Modules
- Intuition pumps
- Notational simplicity
- Notational mimesis
- Notational consistency
- Pictures

Rational Choice?

- What do we mean by rational choice? Not much!
  - Individual have preference
  - faced with a choice between two alternatives he can say whether he does not prefer one to the other or is indifferent between them – complete preferences
  - faced with three choices, x, y, and z, and if the individual prefers x to y and y to z it can't be the case that he prefers z to x – transitive preferences
- That is it, rationality doesn't have any substantive content.

Alternatives & Choices

- Choices/alternatives: $A = \{a_1, a_2, \ldots, a_k\}$
- Outcomes: $X = \{x_1, \ldots, x_n\}$
- Mapping: $x : A \to X$ (assume certainty)
  - x_i is feasible if there exist $a \in A$ such that $x(a) = x_i$

.

Preference Relation

- preferences are a binary relation R on X
  - xRy means x is is weakly preferred to y, i.e., x is at least as good as y
  - xPy iff xRy and $\sim yRx$
  - xIy iff xRy and yRx
- R is sometimes written $\succeq$

Alternatives & Choices

- A rational individual picks $x \in X$ s.t. xRy for all $y \in X$.
  - The question is: Is there such an alternative?
  - Rephrasing the question:

Definition

Given a set X and a weak preference relation R on X, the maximal set $M(R, X) \subset X$ is defined $M(R, X) = \{x \in X | xRy \ \forall y \in X\}$.

The Maximal Set

- Is $M(R, X) = \emptyset$?
- Turns out that if three conditions are satisfied then $M(R, X) \neq \emptyset$:
  - if R is complete: if $x, y \in X$ then xRy, yRx, or both
  - if R is reflexive: if for all $x \in X$, xRx.
  - if R is transitive: if for all $x, y, z \in X$ such that xRy and yRz then xRz.

The Maximal Set

- substance of preferences?
- only logic conditions for individuals to make rational choices
- we normally use a different representation of preferences in our work, i.e., we construct utility functions.

Definition

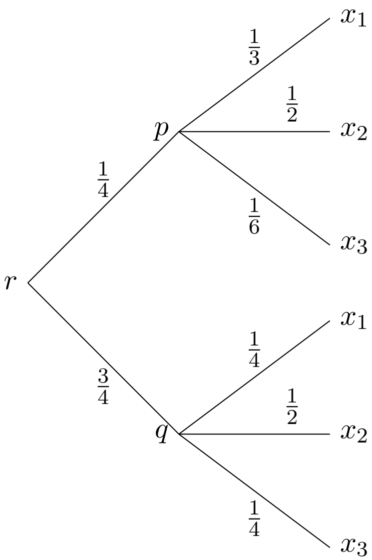
Given X and R on X, the utility function $u : X \rightarrow \mathbb{R}^1$ represents R if $u(x) \geq u(y)$ iff xRy for all $x, y \in X$.

- it follows that $u(x) > u(y)$ iff xPy and $u(x) = u(y)$ iff xIy
- we can then show that $M(X, R) = argmax_{x \in X}\{u(x)\}$.
- **important**: utility functions are derived from preferences, not the other way around. People don't have utility functions!
- utilities generally don't have meaning

The Maximal Set

- what if actions don't lead to **certain** outcomes?
  - "states" of the world: $S = \{s_1, \ldots, s_K\}$
  - probability of state k equals p_k
  - $\sum_{i=1}^{K} p_k = 1$
- before: $x(a_i) = x_i$
- now: $x(a_i, s_1) = x_i$ while $x(a_i, s_2) = x_j$
- actors choose among lotteries

Compound Lotteries



The von Neuman-Morgenstern theorem

The von Neuman-Morgenstern theorem tells us that uncertainty doesn't prevent us from using utility representation if:

1. R on P is complete and transitive.
2. If compound lotteries can be reduced: $pI[\alpha p + (1 - \alpha)p]$
3. Continuity – if we mix two lotteries using a scalar there is one cutpoint.
4. Independence – the above. Don't have to take account of outcomes that occur with same probability.

The von Neuman-Morgenstern theorem

The von Neuman-Morgenstern theorem tells us that then there exists a utility function such that:

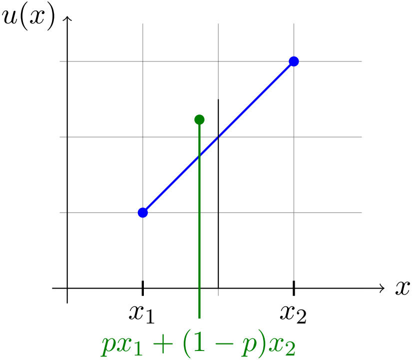
$$EU(p_i) = p_{i1}u_1 + p_{i2}u_2 + \ldots + p_{iJ}u_J = \sum_{j=1}^{J} p_{ij}u_j \tag{1}$$

and

$$p_i R p_j \text{ iff } EU(p_i) \geq EU(p_j) \tag{2}$$

Risk Aversion

- A fair bet: $w = px_1 + (1-p)x_2$
- Risk averse if: $u(px_1 + (1-p)x_2) > pu(x_1) + (1-p)u(x_2)$



Measuring Risk Aversion: Example from Experiment Instructions

We now propose you a series of choices between a fixed amount of money and a lottery. We will pick randomly one of the cases at the end of the session, which will determine the actual outcome for this part.

The lottery will be carried out as follows. We will roll a six-sided die at the end of the session, and you will earn nothing if the die indicates 1, 2 or 3; and earn something (as indicated for each case) if the die indicates 4, 5 or 6.

Please indicate your preferred option in each of the following cases:

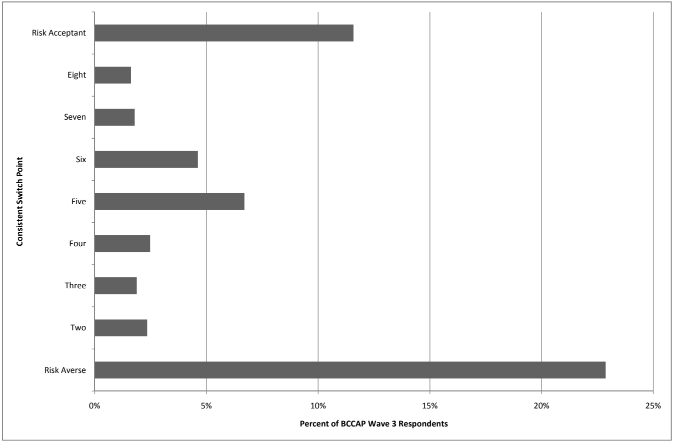
Measuring Risk Aversion: Example from Experiment Instructions

1. Case 1
   1. A: £10 with certainty or
   2. B: £0 if the die shows 1, 2 or 3; £30 if the die shows 4, 5 or 6.
2. Case 2
   1. A: £10 with certainty or
   2. B: £0 if the die shows 1, 2 or 3; £27.50 if the die shows 4, 5 or 6.
3. Case 3
   1. A: £10 with certainty or
   2. B: £0 if the die shows 1, 2 or 3; £25 if the die shows 4, 5 or 6.
4. Case 4
   1. A: £10 with certainty or
   2. B: £0 if the die shows 1, 2 or 3; £22.50 if the die shows 4, 5 or 6.

Measuring Risk Aversion: Example from Experiment Instructions

1. Case 5
   1. A: £10 with certainty or
   2. B: £0 if the die shows 1, 2 or 3; £20 if the die shows 4, 5 or 6.
2. Case 6
   1. A: £10 with certainty or
   2. B: £0 if the die shows 1, 2 or 3; £17.50 if the die shows 4, 5 or 6.
3. Case 7
   1. A: £10 with certainty or
   2. B: £0 if the die shows 1, 2 or 3; £15 if the die shows 4, 5 or 6.
4. Case 8
   1. A: £10 with certainty or
   2. B: £0 if the die shows 1, 2 or 3; £12.50 if the die shows 4, 5 or 6.

Figure: Switching Points for Risk Experiment; UK BCCAP Survey (2009)



Collective Decision Making (Arrow, etc.)

- How do individual-level preferences get aggregated into social (rather than individual) choices?
- is there a method for making claims about preferences or interests of a collectivity, based, presumably, on the preferences or interests of all the individuals of the collectivity or some subset of them
- Arrow's Impossibility Theorem says that there is no way aggregate individual preferences into a rational social preference without violating some basic normative principles

.