Formal Analysis: Lecture 4

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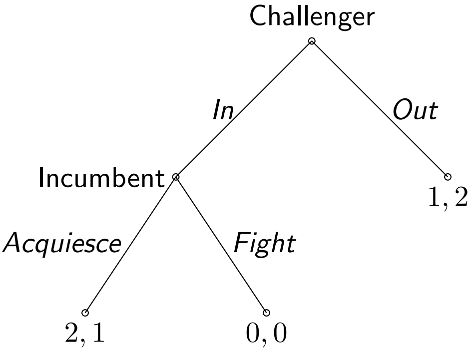
Extensive Games vs. Strategic Games

Strategic Games:
We assume that players make decisions simultaneously (or without observing each other's decisions).
We assume that players can commit to a binding plan of action at the beginning of the game.
Appropriate for static situations.

Extensive Games:
We assume that players make decisions sequentially.
We assume that some players can make up their mind after observing the previous actions of other players.
Appropriate for dynamic situations.

Extensive Form

- The graphical representation is called a game tree



Extensive Form

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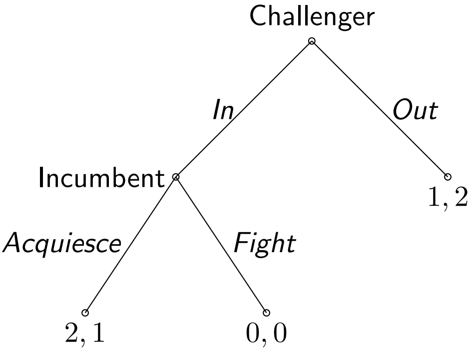
- Extensive form games have four components:



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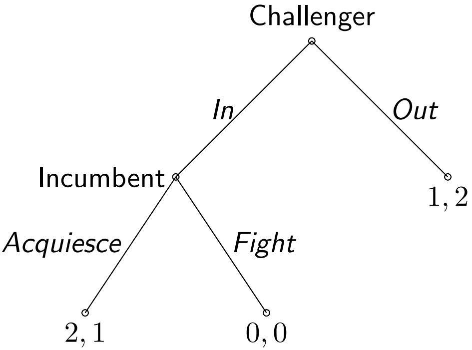

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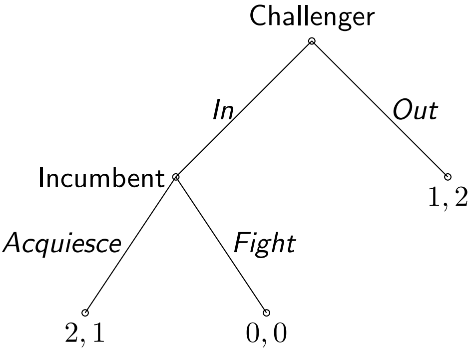

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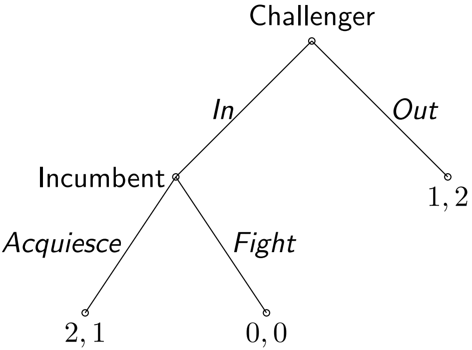

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- Extensive form games have four components:
  - a set of players
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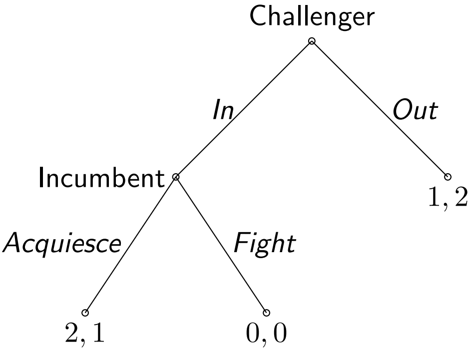
  - a player function that assigns a player to every proper subsequence of some terminal history
  - preferences over the set of terminal histories



Extensive Form

- In this class we will study games with a finite horizon, a finite number of actions and perfect information.

Finite horizon: Terminal histories are not infinitely long. The game has an end.
Finite number of actions: Each player has a limited number of actions to choose from. There is a finite number of terminal histories.
A game with a finite horizon and a finite number of actions is called a finite game.
Perfect information: Each player knows all the actions taken previously (or at least knows what the exact situation is at each point in time). Each player always moves alone at each stage.

- In future classes we will study games with infinite horizons, infinite number of actions and/or imperfect information

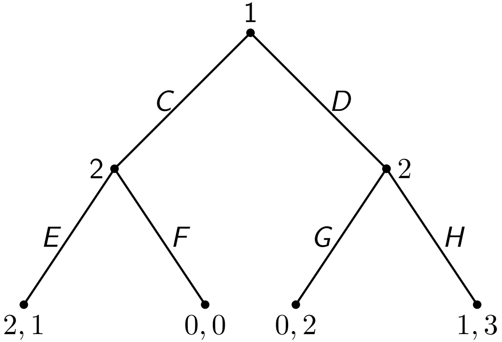
Extensive Form

Solving extensive form games

Strategy: A function that assigns an action to each *history* in which it is the player's turn

Important: A strategy must specify what a player will do in every contingency!

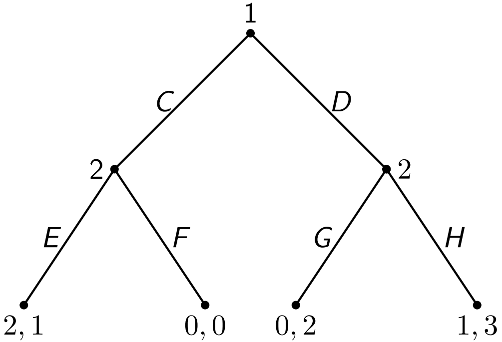
In other words: A strategy is a complete plan of action for every situation in which the player might be called upon to act.



Extensive Form

What are all the possible strategies of player 2?

	Action assigned to history C	Action assigned to history D
Strategy 1	E	G
Strategy 2	E	H
Strategy 3	F	G
Strategy 4	F	H



Nash Equilibrium

- We refer to a profile of strategies as *s*.
- The outcome of game is determined by the actions taken along the path of play specified by the strategies in *s*.
- We refer to outcomes as $O(s)$

Definition (Nash Equilibrium)

The strategy profile s^* is a NE if for all players and every strategy r_i the terminal history $O(s^*)$ is at least as good as those generated by any deviation.

$$u_i(O(s^*)) \geq u_i(O(r_i, s^*_{-i})), \forall r_i \text{ of player i}$$

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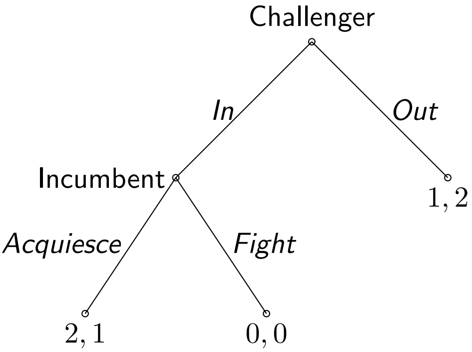
Definition

$$u_i(\alpha^*) \geq u_i(\alpha_i, \alpha^*_{-i}), \forall \alpha_i \in A_i$$

Nash Equilibrium

- Consider the Incumbent vs. Challenger game again



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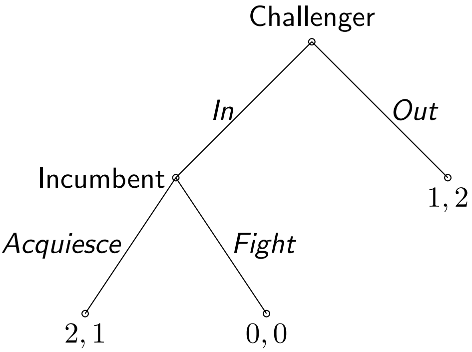
- To find the the NE we need three components:



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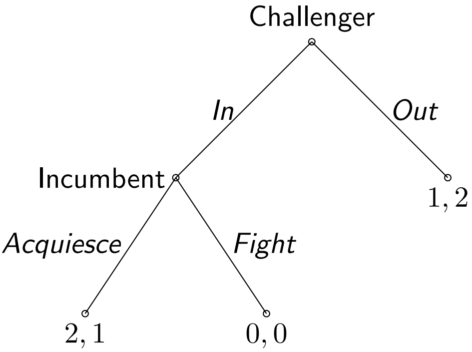

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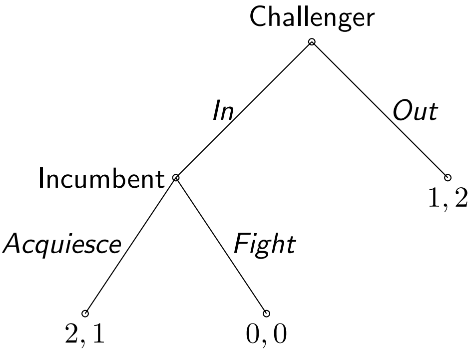

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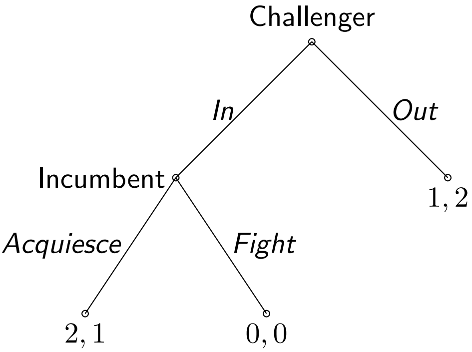
- Consider the Incumbent vs. Challenger game again
- To find the the NE we need three components:
  - the set of players
  - the set of strategies (the actions chosen in each node)
  - preferences over the set of possible outcomes



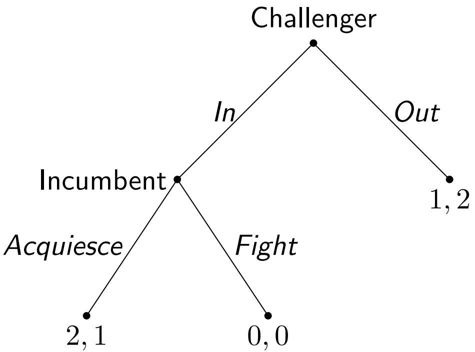
Nash Equilibrium

- Thus, one way to find the NE is to construct a corresponding strategic form game
- Challenger vs. Incumbent game in strategic form:

		Incumbent	
		Acquiesce	*Fight*
Challenger	*In*	$2, 1$	$0, 0$
	Out	$1, 2$	$1, 2$

- This game has two Nash equilibria: (*In*, *Acquiesce*) and (*Out*, *Fight*)

Nash Equilibria



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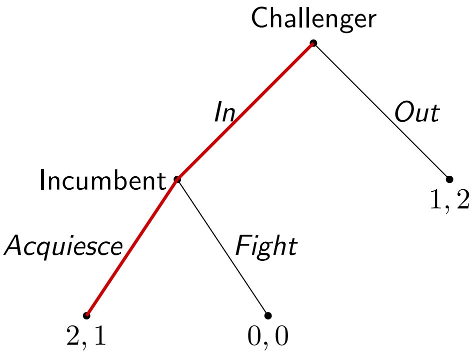

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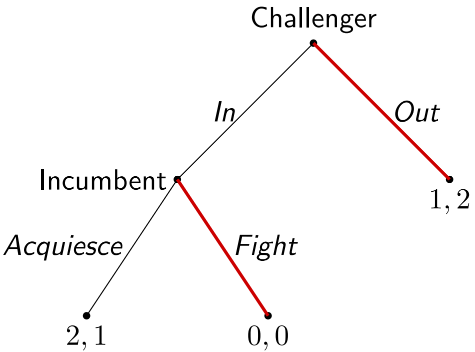

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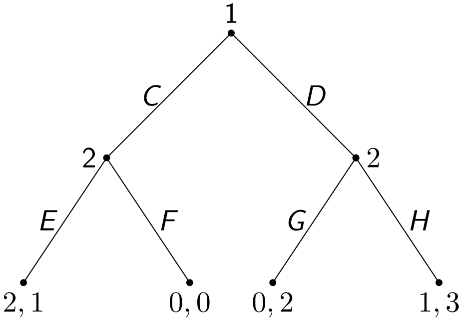

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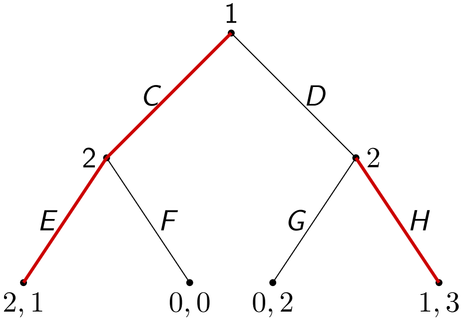

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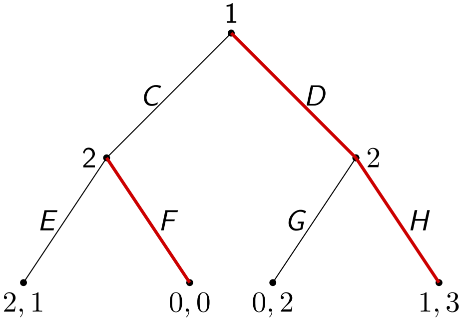
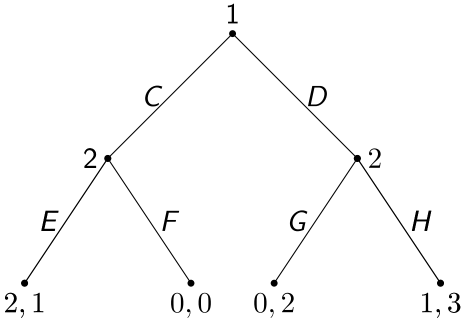


Subgame Perfect Equilibrium

- So the concept of Nash equilibrium makes "too many" predictions. Can we find a different concept that does not predict non-credible threats?
- We start by defining the concept of a **subgame**. A subgame $\Gamma(h)$ is the game following history *h*. The whole game is a subgame of itself. The other subgames are called **proper subgames**.



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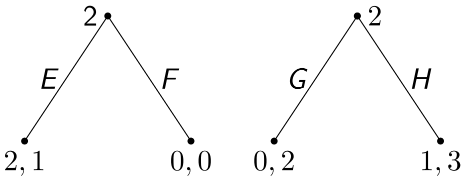
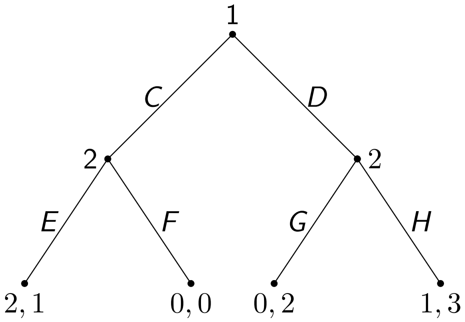


Subgame Perfect Equilibrium

- A more refined equilibrium concept is that of **Subgame Perfect Equilibrium** (SPE)
- A player's actions must be optimal in every subgame of the game.
- Every SPE is a NE but not vice versa.
- A SPE is a strategy profile that induces a NE in every subgame.



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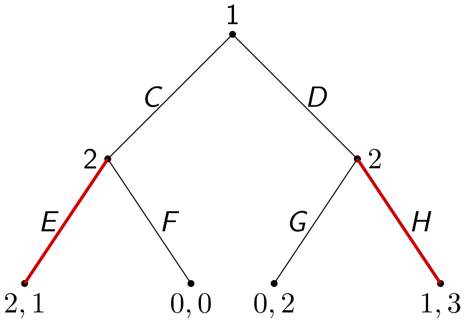

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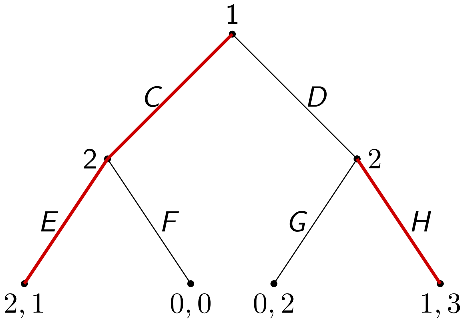
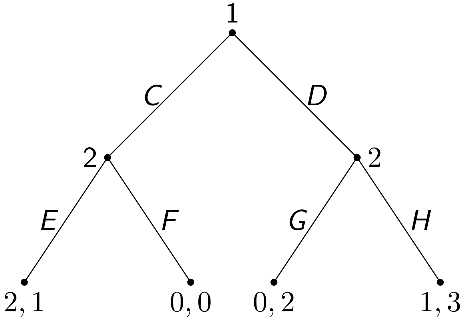


Backwards induction in finite horizon games

- How do we find the SPE?
- Start at the bottom of the tree (the subgames with least length).
- Find the optimal action for each player at that stage of the game
- Construct the **reduced game** that takes the subsequent optimal actions as given. Apply the same procedure to the bottom of that new tree.
- This process is called **backwards induction**.



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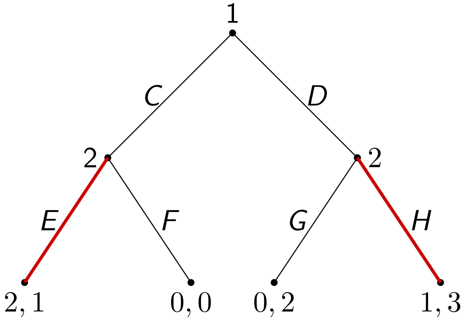

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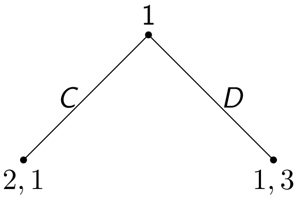

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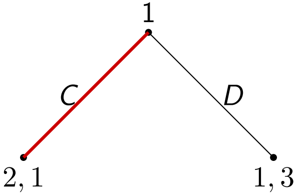

Conclusions

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- But do SPE always exist? Are they difficult to find?

Theorem (Existence of subgame perfect equilibria)

In every extensive game with a finite horizon, a finite number of actions and perfect information there exists at least one SPE.

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Theorem

If an extensive game has a finite horizon and perfect information then its subgame perfect equilibria can be found by backwards induction.