Formal Analysis: Bargaining

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February 22, 2010

Introduction

- Division of resources is at core of politics
- Often focus on two key aspects of equilibria:
  1. Distributional consequences
  2. Efficiency
- Highlight two approaches:
  1. Axiomatic, cooperative: Assume properties that bargaining outcomes should exhibit
  2. Strategic, non-cooperative: Assume particular bargaining protocol

Setting up the bargaining problem

- *Players:* $N = \{1, 2, \ldots, n\}$
- *Outcome space:* X
- *Status quo:* $\{R\} \in X$
- *Utilities:* von Neumann-Morgenstern (Bernoulli) payoff functions u_i for $i \in N$
- *Possible payoffs:* $U = \{(v_1, v_2, \ldots, v_n) : u_i(x) = v_i$ for some $x \in X$ and $i \in N\}$. Let v denote $(v_1, v_2, \ldots, v_n)$.
- *Reservation values:* $r = (u_1(R), u_2(R), \ldots, u_n(R))$, where $r \in U$

Here, Bernoulli payoffs imply convex U as well as u_i that are unique up to a positive affine transformation

What are bargaining problems and solutions?

- *Bargaining problem:* (U, r), where U is compact and $\exists$ v such that $v_i > r_i \ \forall \ i$
- *Bargaining solution:* A function $f : (U, r) \rightarrow v$

Axioms

- *Pareto efficiency (PAR):* Let v and v' be members of U. If $v_i \geq v_i' \; \forall \; i$ and $\exists \; i$ s.t. $v_i > v_i'$, then a bargaining solution does not assign v' to a bargaining problem (U, r).
- *Symmetry (SYM):* Let v be a member of U if and only if v' is also a member of U, where v' is a permutation of v, and let $r_i = r_j \; \forall \; i, j \in N$. Then a bargaining solution v^* satisfies $v_i^* = v_j^* \; \forall \; i, j \in N$.

Axioms II

- *Invariance to equivalent payoff representations (INV):* Let U' consist of members v' with elements $v'_i = a_i + b_i v_i$ for $i \in N$, where v (of which v_i is an element) is a member of U, and a_i and b_i are scalars with $b_i > 0$. Let r' be the n-tuple with element $r'_i = a_i + b_i r_i$. If a bargaining solution assigns v to (U, r), then it assigns $(a_1 + b_1 v_1, a_2 + b_2 v_2, \ldots, a_n + b_n v_n)$ to (U', r').
- *Independence of irrelevant alternatives (IIA):* Let (U, r) and (U', r') be bargaining problems, where U' is a subset of U and $r = r'$. If the bargaining solution assigns an agreement to (U, r) that is also in U', then the bargaining solution assigns the same agreement to (U', r').

Nash bargaining solution

Proposition (Nash bargaining solution)

A bargaining solution satisfies PAR, SYM, INV, and IIA iff it is given by $\arg\max_v \prod_{i=1}^{n}(v_i - r_i)$.

In other words, choose the outcome that maximizes the product of everyone's utilities.

Outline of proof

Sufficiency ("if"):

- Properties of hyperboloid given by $\prod_{i=1}^{n}(v_i - r_i) = c$, where c is a constant, imply that PAR and SYM are satisfied
- Maximization ensures that IIA is satisfied
- Bernoulli payoffs imply that INV is met

Outline of proof II

Necessity ("only if"):
Outline for two players; see Humphreys, p. 183, for proof with n players

- Normalize bargaining problem (U, r) with some Pareto efficient member (z_1, z_2) so that $r' = (0, 0)$ and $z_1 = z_2 = \frac{1}{2}$. Call this new bargaining problem (U', r').
- Suppose it is possible to construct yet another bargaining problem (U'', r''), such that the Pareto frontier of U'' consists of payoff pairs $(v_1, 1 - v_1)$ (which means $(\frac{1}{2}, \frac{1}{2})$ is Pareto efficient), U'' is symmetric, and $U' \in U''$.

Outline of proof III

- Now any bargaining solution that satisfies SYM and PAR assigns $(\frac{1}{2}, \frac{1}{2})$ to (U'', r''). But then it must also assign $(\frac{1}{2}, \frac{1}{2})$ to (U', r') if it satisfies IIA.
- Finally, U' is an affine transformation of U, so any bargaining solution that satisfies INV and assigns $(\frac{1}{2}, \frac{1}{2})$ to (U', r') must assign (z_1, z_2) to (U, r).
- Lastly, we need to find (z_1, z_2) such that $U' \in U''$. But that's just $\arg\max_{(v_1, v_2)}(v_1 - r_1)(v_2 - r_2)$.

Implications

- Each individual's share increasing in own reservation value and declining in others'
- If utilities symmetric, receive reservation value plus equal share of surplus
- But why should bargainers cooperate? Bargaining protocols, institutions matter

Alternating offers as a variant of the "ultimatum game"

- *Players:* $\{1, 2\}$
- *Terminal histories:* Finite sequences $(x^1, N, x^2, N, \ldots, x^t, Y)$ for $t \geq 1$ and every infinite sequence $(x^1, N, x^2, N, \ldots)$, where each $x^i = \{x_1, x_2\}$ and $x_1 + x_2 \leq 1$
- *Player function:* $P(\varnothing) = 1$, and

$$P(x^1, N, x^2, N, \ldots, x^t) = P(x^1, N, x^2, N, \ldots, x^t, N) = \begin{cases} 1 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd.} \end{cases}$$

- *Preferences:* For $i \in \{1, 2\}$, player i's payoff to terminal history $(x^1, N, x^2, N, \ldots, x^t, Y)$ is $\delta_i^{t-1} x_i^t$, where $0 < \delta_i < 1$, and payoff to infinite terminal history is 0

How do we solve this game?

- We can't use backward induction because some of the terminal histories involve infinite sequences
- Note that game is *stationary:* Any subgame that starts with a proposal by player i is identical to any other subgame that starts with such a proposal
- Perhaps we can find SPNE in stationary strategies (but note that such a SPNE need not be unique or even exist)

A conjecture

- Intuitively, what might such an equilibrium look like?
  1. Each player will accept sufficiently high offers and reject all others
  2. Since all offers are immediately accepted if the game is finite, let's suppose that this is the case here as well
  3. Each player will never offer more than necessary in order for a proposal to be accepted
  4. Players will always propose to divide the entire pie
- This implies that player 1 proposes x^* and accepts proposal y if and only if $y_1 \geq y_1^*$, and player 2 proposes y^* and accepts proposal x if and only if $x_2 \geq x_2^*$ (see Osborne, p. 470)

A conjecture II

- Since payoffs are discounted by δ_i for each period of delay, the minimum offer that 2 will accept is then $x_2^* = \delta_2 y_2^*$. Similarly, $y_1^* = \delta_1 x_1^*$.
- Since neither player will waste any resources, we have $x_1^* = 1 - x_2^*$ and $y_1^* = 1 - y_2^*$, and hence:

$$x_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$$
$$y_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$$

More details

Equilibrium statement

Proposition (SPNE of bargaining game with alternating offers)

This game has a unique SPNE, where player 1 always proposes x^* and accepts a proposal y if and only if $y_1 \geq y_1^*$, player 2 always proposes y^* and accepts a proposal x if and only if $x_2 \geq x_2^*$, where

$$x^* = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \right)$$
$$y^* = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2} \right)$$

One-deviation property: Proposition

How do we prove this proposition? Using the one-deviation property!

Proposition (One-deviation property of SPNE)

A strategy profile σ^* in an extensive game with either a finite horizon or an infinite horizon with a discount factor less than 1 is a SPNE if and only if there is no stage of the game at which any player can gain by changing her strategy at that stage, given all other players' strategies and her own strategy at all other stages.

One-deviation property: Outline of proof

- See also Humphreys, p. 143f
- First, consider finite games.
- Necessity ("only if") follows from the definition of SPNE, which requires that no deviation is optimal.

One-deviation property: Outline of proof II

- For sufficiency ("if"), consider a rival strategy σ' that is a profitable deviation from σ^*, but is different from the latter after as few histories as possible.
- Let's focus on the longest history h^* after which the strategies differ. For the subgame starting at h^*, the strategies differ *only* at h^*.
- Also note that the deviation at h^* must be profitable, because otherwise σ' would not differ from σ^* after as few histories as possible.
- But this implies that if a profitable deviation σ' exists, then a subgame exists in which only a deviation at the beginning of that subgame is profitable.
- In turn, if a deviation only at the beginning of the subgame is not profitable in any subgame, then no deviation is profitable at all.

One-deviation property: Outline of proof III

- For infinite games with $\delta < 1$, any payoff differences in the sufficiently distant future become arbitrarily small. But this means that any player will be indifferent between σ^* and σ' after some sufficiently large T and we can proceed as in the proof for finite games with T time periods.

Outline of proof for SPNE

- Now, back to proving the SPNE in the bargaining game of alternating offers
- There are just two types of subgames to consider (thanks to stationarity): Those that start with offers and those that start with a response
- First, consider 1's offer decision, which yields x_1^* in the proposed equilibrium; less than x_1^* if he offers more than x_2^* (which 2 accepts); and $\delta y_1^* < x_1^*$ if he proposes less than x_2^* (which 2 rejects).
- Second, 1's response decision yields y_1^* in equilibrium and $\delta_1 x_1^* = y_1^*$ if he deviates.
- Proof for player 2 proceeds symmetrically
- For proof of uniqueness, see McCarty and Meirowitz (p. 223f)

Properties of equilibrium

- Efficiency: No delay
- Patience pays
- First-mover advantage

How does this relate to Nash bargaining solution?

- Suppose $\delta_1 = \delta_2 = \delta$. Then player 1's payoff becomes

$$\frac{1-\delta}{1-\delta^2} = \frac{1-\delta}{(1-\delta)(1+\delta)} = \frac{1}{1+\delta}$$

- As δ approaches 1, this payoff approaches $\frac{1}{2}$, in line with the Nash bargaining solution (see McCarty and Meirowitz, p. 225)
- Limitation of alternating offers game: Requires unanimous consent

Setup of game

- N players, of which $n = (N+1)/2$ must vote in favor of proposal for it to pass
- Random recognition: In each period, each player gets to make proposal with probability $1/N$
- Proposal given by $(x_1, x_2, \ldots, x_N)$, with $\sum_{i \in N} x_i \leq 1$
- If proposal rejected, discount by factor δ, and new proposer is selected

Equilibrium description

- Many SPNE, so focus on equilibria that are stationary, symmetric, and exclude weakly dominated strategies
- Proposer must give δV to $n-1$ players (and 0 to everyone else), where V is continuation value
- Proposer keeps $z = 1 - (n-1)\delta V$
- What is V? $\frac{z}{N} + \frac{n-1}{N}\delta V = \frac{1}{N}$
- Hence, $z = 1 - \delta \frac{n-1}{N} = 1 - \delta \frac{N-1}{2N}$

Implications for proposer's advantage

- As before, decreasing in δ
- Increasing in N
- Decreasing in supermajority requirements (higher n)
- Reduced with open rule

Algebra for bargaining game of alternating offers

We substitute $x_2^* = \delta_2 y_2^*$ in $x_1^* = 1 - x_2^*$, and $y_1^* = \delta_1 x_1^*$ in $y_2^* = 1 - y_1^*$, which leaves us with

$$x_1^* = 1 - \delta_2 y_2^*$$
$$y_2^* = 1 - \delta_1 x_1^*.$$

Further substitution for y_2^* yields

$$x_1^* = 1 - \delta_2(1 - \delta_1 x_1^*)$$
$$x_1^* - \delta_1 \delta_2 x_1^* = 1 - \delta_2$$
$$x_1^* = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

We similarly substitute to obtain y_1^*, x_2^*, and y_2^*.

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