Games with Imperfect Information I: Bayesian games

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March 1, 2010

Review: Normal form games

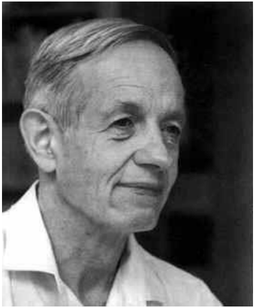
Normal form game is three things:

- A set of players $\mathcal{N}$
- A set of actions available to each player, $\{A_i\}_{i\in\mathcal{N}}$ (with $A = \times_{i\in\mathcal{N}} A_i$)
- Preferences of each player, depending on the actions of all: $\{u_i\}_{i\in\mathcal{N}}$ with

$$u_i : A \to \mathbb{R}$$

Review: Normal form games

And our *equilibrium* concept was



Nash Equilibrium!!

A profile of actions a^* is a (pure strategy) Nash Eq (NE) iff

$$u_i(a^*) \geq u_i(a_i, a^*_{-i})$$

for all a_i, for all $i \in \mathcal{N}$

Motivation

Up till now, strategic situations are ones in which *everyone knew everything* (in equilibrium).

- structure of game
- actions available
- payoffs of others

They were games of *complete* and *perfect* information....

- *complete* information: structure of the game, payoffs of all
- *perfect* information: how all act

Motivation 2

BUT!!
Would like to relax this to study interactions in which actors do not “know everything”:

- War: don’t know strength of opponent/ don’t know opponent’s utilities (like to fight?)
- Candidate competition: don’t know public preferences/ don’t know opponent’s “abilities” (war-chest, etc.)

Definition

[Note: departs slightly from Osborne Def. 279.1]
A Bayesian game in normal form is:

- a set of **players**
- a set of **states** (of nature), $\omega \in \Omega$ (assume countable for now – easily generalized)
- a set of **actions** for each player
- a set of **signals** for each player (also called '**private information**'), $\mathcal{T}_i$ or "type space"
- von Neumann-Morgenstern **utility** for each player. ie Bernoulli payoffs that depends on *others' actions* and the *state* of the world: $u_i(\sigma, \omega) \to \mathbb{R}$

and....
beliefs for each player (almost... more on this later)
-if agents don't know about the environment, how are they to act?
-what do we assume about how agents form and/or update beliefs?

Definition, Intuition

Think of a Bayesian game (for now) as:
-Each player has an 'idea' about the world: whatever an agent doesn't **know** about, it has **beliefs** about
-Get a signal (private information): something you know that others don't
-Update beliefs: now I know something more than I did at the beginning of the game, namely my own private info. How does that change my beliefs?
–BAYES' RULE
-Take action (perhaps probabilistically): maximize expected utility, given all info + beliefs you have.

Details

- Strategies for normal form games: (new) $\sigma_i : \mathcal{T}_i \to \Delta S_i$ – now depends on *type* (private info)
- Payoffs: depend on actions of all, and on state of nature. Bernoulli payoffs $u : S \times \Omega \to \mathbb{R}$.
- Expected utility (depends on type (signal) and actions): $U_i(\sigma, t_i) = \sum_{\omega \in \Omega} Pr[\omega, t_{-i} | t_i] u_i(\sigma_i(t_i), \sigma_{-i}(t_{-i}), \omega)$

Bayes' Rule

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)} \tag{1}$$

If $C_1, \ldots, C_N$ are events that partition the whole space ie, $\sum Pr(C_n) = 1$, $C_j \cap C_k = \emptyset$ and $Pr(C_n) > 0$ for all n, then:

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{\sum_{n=1}^{N} Pr(B|C_n)Pr(C_n)} \tag{2}$$

Direct application of NE → **Bayesian Nash Eq.**

Definition

A strategy profile σ^* is a Bayesian Nash Equilibrium of a Bayesian strategic form game if

$$\sum_{\omega\in\Omega} Pr[\omega, t_{-i}|t_i] u_i(\sigma_i^*(t_i), \sigma_{-i}^*(t_{-i}), \omega) \geq \sum_{\omega\in\Omega} Pr[\omega, t_{-i}|t_i] u_i(\sigma_i'(t_i), \sigma_{-i}^*(t_{-i}), \omega)$$

for all i, for all σ_i'

Provision of a Public Good (modified Palfrey-Rosenthal 1988)

- n players
- Actions = contribute or not, $A_i = \{0, 1\}$ for all i

$$u_i(1, a_{-i}) = \begin{cases} 1 - c_i & \text{if } \sum a_i \geq k \\ \text{-c}_i & \text{otherwise} \end{cases}$$

$$u_i(0, a_{-i}) = \begin{cases} 1 & \text{if } \sum a_i \geq k \\ 0 & \text{otherwise} \end{cases}$$

- Private information: $c_i \sim \mathcal{U}[0, 1]$.

Provision of a Public Good, $k = 1$

Consider symmetric eq. (all c_i employ same strategy) Now:

- Asymmetric Eq: $a_i = 1, a_{-i} = 0$ is an eq for any i.
- Cut point strategies: $u_i(1, a_{-i}) = 1 - c_i$, $u_i(0, a_{-i}) = p_i$, so best response function looks like "contribute if $c_i > 1 - p_i$." So focus on strats $\hat{c}_n$ such that "contribute if $c_i > \hat{c}_n$." What if $c_i = \hat{c}_n$?
- Others contribute with prob $\hat{c}_n$. Why?
- $\rightarrow$ Prob. that no one else contributes is $(1 - \hat{c}_n)^{n-1}$.
- Contribute if $E[u_i(1, \cdot)] > E[u_i(0, \cdot)]$.
- Indifference requires $(1 - \hat{c}_n)^{n-1} = \hat{c}_n$.

Note:

- $\hat{c}_n$ decreasing in n. Why?

Provision of a Public Good, $k > 1$

- Let x_{-i} be realized number of other contributions, $x_{-i} = \sum_{j \neq i} a_j$
- Net utility: $u_i(1, x_{-i}) - u_i(0, x_{-i}) = Pr[x_{-i} = k-1] - c_i$
- ex ante: $Pr[x_{-i} = k-1] = \binom{n-1}{k-1} \hat{c}_n^{k-1}(1-\hat{c}_n)^{n-k}$. Why?
- Again, indifference implies $\binom{n-1}{k-1} \hat{c}_n^{k-1}(1-\hat{c}_n)^{n-k} = \hat{c}_n$
- Let $\Pi(\hat{c}_n) = \frac{\binom{n-1}{k-1} \hat{c}_n^{k-1}(1-\hat{c}_n)^{n-k}}{\hat{c}_n}$
- Indifference implies $\Pi(\hat{c}_n) = 1$
- ...
- (Approximately) $\hat{c}_n = \frac{k-2}{n-2}$, provided $2 < k < n$

Uncertainty of Candidate Preferences (M&M pg. 164)

Two policy motivated candidates, ideal points (in 1-D) unknown. One median voter.
Set up:

- $\theta_1 \in \{0, 1/2\}, \theta_2 \in \{1/2, 1\}$
- $u_i(x) = -(\theta_i - x)^2$, $x =$ implemented policy
- median voter's ideal point $\sim \mathcal{U}[0,1]$
- strategies: $s_1(\theta_1) : \{0, 1/2\} \to [0, 1/2]$ (for simplicity) and vice versa
- Assume 1 uses $s_2(1/2) = a$ and $s_2(1) = b$. What about $\theta_1 = 1/2$?
- $s_1 = 1/2$ dominates any $s_1 < 1/2$. Why?
- So $s_1(1/2) = 1/2$ and $s_2(1/2) = 1/2$

- What about s_1 when $\theta_1 = 0$?
-
$$max -s_1^2\left(\frac{s_1+1/2}{4}+\frac{s_1+b}{4}\right)-\frac{.5^2}{2}\left(1-\frac{s_1+1/2}{2}\right)$$
$-\frac{b^2}{2}\left(1-\frac{s_1+b}{4}\right)$
- (whew)
- Differentiate this and set equal to zero (some more math)
- $b = \frac{11}{7} - \frac{\sqrt{106}}{14} \simeq 0.836$
- SO: $s_2(1/2) = 1/2, s_2(1) \simeq 0.836 \rightarrow$
- when cand. prefs. uncertain-> more divergent platforms than when cand. prefs are known! Why? candidates policy motivated -> would rather loose to a moderate than to an extremist -> dampens incentives for extreme candidates to moderate.

Types of Uncertainty

What can agents be uncertain of?

- Payoffs (own or others)
- Actions taken by others

Harsanyi!!
-Any game of *incomplete* information can be transformed into a game of *imperfect* information (uncertainty about history of play)!
Things to think about..

- Set of actions?
- Number of players?