Games with Imperfect Information II: Extensive Form Games, Examples and Applications

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Today

- Clarify indifference condition from last time
- Extend settings of uncertainty
- Extensive form games with uncertainty
- Representation, game trees
- Solution concepts: PBNE, Sequential Equilibria
- Applications

Clarify indifference condition:

If agent i is indifferent between two actions, a and b, then the *expected utility* is the same:

$$u_i(a, \cdot) = u_i(b, \cdot).$$

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  $+(-c_i)(1 - Pr[\sum_{j \neq i} s_j \geq k - 1])$
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- Reduces to: $(1 - \hat{c}_n)^{n-1} = \hat{c}_n$ ie expected utility not just about probability of 'being pivotal' - also about **payoff** (ie consequence) *if your action matters*

Extending Last Time

Uncertainty + sequential moves
-last time: uncertainty (of payoffs, of state of nature, etc.) in *static* games (ie normal form games)

Definition

An **extensive form game** (in full generality) is

- Set of **agents**, N
- Set of **histories** H (think: nodes in a tree) with: (1) initial history $H^0 = \emptyset$ and (2) terminal histories H^T
- **Order** of moves (think: who moves when): $p(h) : H \setminus H^T \rightarrow N$
- **Actions** available to each player: $A(h)$ that player "at" history h (this player is given by $p(h)$) may take
- **Information sets** $I \subset H \setminus H^T$: partition H (think: if player $i = p(h)$) is called to move at information set I_h, s/he doesn't know which node in I_h she is at
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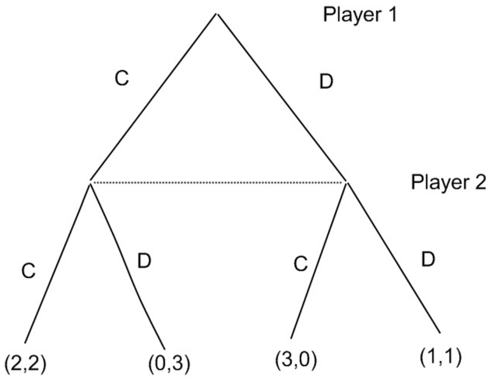
- (see M&M pg. 173 or Os pg 314 for formal requirements for "tree" ie no cycles, two nodes don't lead to same node, etc.).

Illustration 1: Prisoner's Dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$



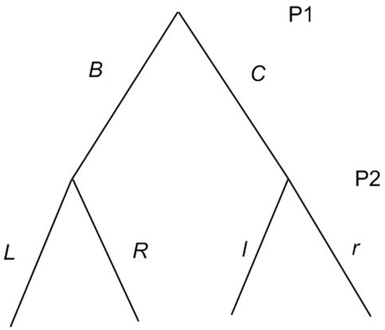
Strategies

Let H_i be the set of histories for which player i is called to act: $H_i = \{h|p(h) = i\}$.

Definition

A strategy (for player i) in an extensive form game is a map $s_i : H_i \to A(h)$ such that $s_i(h) = s_i(h')$ if h and h' are in the same info set.

Illustration 2: strategies



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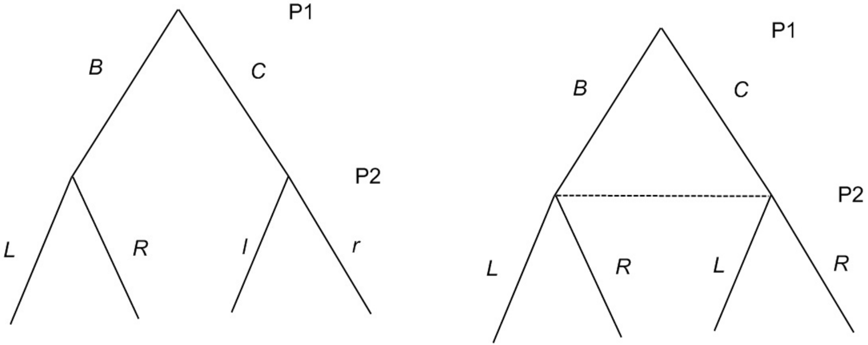

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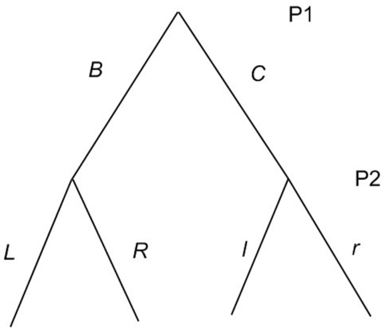

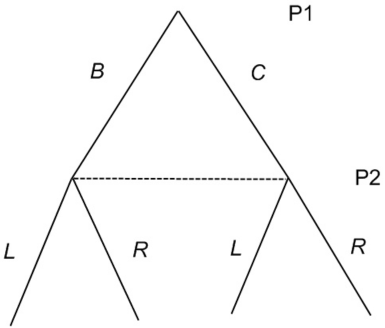

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- Perfect Bayes Nash Eq (PBNE): Require actions and beliefs to be consistent at all sub- games
  - Sub-game = starts at a *node* (and not an info set)
- Sequential Equilibria: require actions and beliefs to be a stronger notion of consistent (PBNE only applies to info sets reached with positive probability, might want something stronger)

Sol. Con. 1: Perfect Bayes Nash Equilibria

A PBNE is two things:

- Strategy: for each info set, a (probabilistic) prescription of what do do next and
- Beliefs: probability distribution over *each* info set. Ie for each node I could be at, what probability that I am actually at that node?

such that

- Strategies are 'rational' – strats are optimal, given beliefs
- Beliefs are consistent with the strategy profile (ie obtained via Bayes rule – when applicable – and strategy profile)

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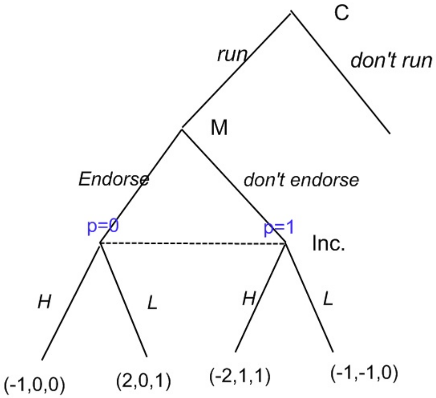
-Challenger runs or not
-Media endorses or not and Incumbent puts in High Effort or Low Effort
-Preferences:

- C wants to enter only if (i) media endorses and (ii) Inc. low effort;
- Media wants to endorse C only if Inc is low effort;
- Inc. wants to exert effort only if C not endorsed

Illustration: PBNE, Cont.



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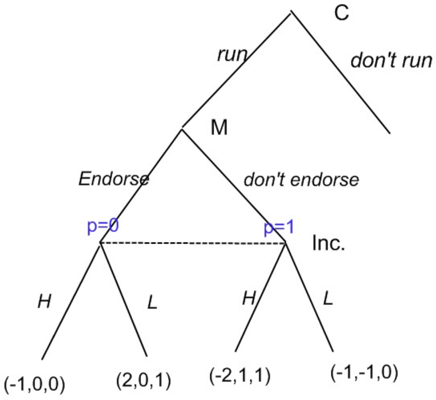

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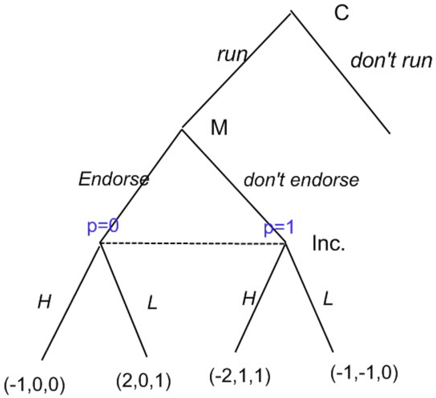

Consider strategy: (s_C =don't run ; s_M =endorse if C runs and; $s_{Inc} = H$ if M endorses) and beliefs Pr[Media endorses Challenger] $= 0$ This is a PBNE. This is not sub-game perfect. Why?

Sol. Con. 2: Sequential Equilibria

Definition

For a finite extensive form game with imperfect information, a *sequential equilibrium* is a pair (β, μ) such hat (1) β is sequentially rational and (2) there exist a sequence of fully mixed strategies and beliefs converging to β, μ that are PBNE[1]

[1] "eventually" – see def. 8.6 in M& M (pg 238) if you are really interested in the technical condition for "eventually."

Example: Strategic Information Transmission

Set up: Legislature (L) and beaurocracy (B) with different preferences.
B observes some information, $t \in [0,1]$ (think "(true) good policy"), and communicates some message r to Leg.
Leg. has prior beliefs $t \sim \mathcal{U}[0,1]$ and takes some action y (sets policy, enforcement level, etc.)
Payoffs:

$$u_B(y,t) = -(y-(t-b))^2$$

$$u_L(y,t) = -(y-t)^2$$

Picture

Example, Cont.

Look for PBNE.

- Full information transmission: can we have $r(t) = t$??
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  - (what does Leg. believe after receiving $c' \neq c$? Does it matter? Assume Leg. beliefs are constant.)
  - If Leg. beliefs are constant, B cannot influence Leg. action $\rightarrow r(t) = c$ a best reponce

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Look for Eq in which $r(t) = \begin{cases} r_1 & \text{if } 0 \leq t < t_1 \\ r_2 & \text{if } t_1 \leq t \leq 1 \end{cases}$

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- r_1 optimal when $u_B(t_1/2, t) \geq u_B(\frac{1+t_1}{2}, t)$ for t/get_1 (similar for r_2)

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- $\rightarrow u_B(t_1/2, t_1) = u_B(\frac{1+t_1}{2}, t_1)$
- $\rightarrow$ in state t_1, the bearuocrat is indifferent $\rightarrow t_1 + b$ is midway between $t_1/2$ and $\frac{1+t_1}{2}$
- $t_1 = \frac{1}{2} - 2b$
- Note: for the conjectured Eq to exist, we need $t_1 > 0$ so that $b < \frac{1}{4}$.