Formal Analysis: Lecture 3

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Illustrations: Hotelling's Model of Electoral Competition

- Two Candidate Election
  - Candidates x_1 and x_2
  - Suppose $x_2 < m$
  - If $x_1 < x_2$ then x_2 attracts all votes $> \frac{1}{2}(x_1 + x_2)$ thus winning>50 percent of votes

Illustrations: Hotelling's Model of Electoral Competition

- Best response function for Player 1 ($B_1(x_2)$)
    - $x_1 : x_2 < x_1 < 2m - x_2$ if $x_2 < m$
    - m if $x_2 = m$
    - $x_1 : 2m - x_2 < x_1 < x_2$ if $x_2 > m$
- Best response function for Player 2 ($B_2(x_1)$)
    - $x_2 : x_1 < x_2 < 2m - x_1$ if $x_1 < m$
    - m if $x_1 = m$
    - $x_2 : 2m - x_1 < x_2 < x_1$ if $x_1 > m$
- Unique Nash Equilibrium is where both candidates choose the position m

Matching Pennies Revisited

		Actor 2	
		$Heads$	$Tail$
Actor 1	$Heads$	$1, -1$	$-1, 1$
	$Tails$	$-1, 1$	$1, -1$

- How would we play this game?
- Suppose actor 2 chooses H with probability $\frac{1}{2}$. What is actor 1's best response – including mixed strategies?
- Suppose actor 1 chooses H with probability p
    - Probability of winning: $\frac{1}{2}p + \frac{1}{2}(1 - p) = \frac{1}{2}$
    - Probability of losing: $\frac{1}{2}p + \frac{1}{2}(1 - p) = \frac{1}{2}$

Matching Pennies Revisited

$$\begin{array}{ccc|c|c|} & & \multicolumn{2}{c}{\text{Actor 2}} \\ & & Heads & Tail \\ \cline{3-4} \text{Actor 1} & Heads & 1,-1 & -1,1 \\ \cline{3-4} & Tails & -1,1 & 1,-1 \\ \cline{3-4} \end{array}$$

- Other equilibria?
  - Suppose player 2 mixes with probability q
    - The probability that 1 wins equals: $pq + (1-p)(1-q)$
    - & loses with probability: $p(1-q) + (1-p)q$
    - simplifies to $1 - q + p(2q - 1)$ and $q + p(1 - 2q)$
  - If $q < \frac{1}{2}$ then Player 1 chooses Tails

Notes re: Mixed Strategy Equilibrium

Now if $q < \frac{1}{2}$ then $2q - 1$ is negative and choosing a higher p will reduce the likelihood of winning. Thus, player 1 would choose $p = 0$. Now suppose $q > \frac{1}{2}$ – then $2q - 1$ is positive and choosing a higher p increases the probability of winning, i.e, player 1 will choose $p = 1$. Thus, there are no other mixed strategy equilibria.

Mixed Strategies: Definitions

Definition

A strategic game (with vNM preferences) consists of $\langle N, \{A\}_i, \{u\} \rangle$ where preferences u_i are represented by expected value payoff functions (Bernoulli).

Definition

A mixed strategy of a player in a strategic game is a probability distribution over the player's actions

Let α denote a profile of mixed strategy, $\alpha_i(a_i)$ is the probability assigned by player i's mixed strategy to a_i.

Nash Equilibrium in Mixed Strategies

Equilibrium and best responses are defined in the same manner as before.

Definition (Nash Equilibrium in Mixed Strategies)

$u_i(\alpha^*) \geq u_i(\alpha_i, \alpha^*_{-i})$, $\forall \alpha_i \in A_i$ and $\forall i \in N$

The mixed strategy profile α^* is a mixed strategy profile if and only if α^* is in $B_i(\alpha^*_{-i}), \forall i \in N$.

Nash Equilibrium in Mixed Strategies

Consider a two player strategic game. Assume P1 plays T with probability p and P2 L with q.

		Actor 2	
		L	R
Actor 1	T	pq	$p(1-q)$
	B	$(1-p)q$	$(1-p)(1-q)$

P 1's expected payoff is then:

$$pqu_1(T,L)+p(1-q)u_1(T,R)+(1-p)qu_1(B,L)+(1-p)(1-q)u_1(B,R)$$

or $p[qu_1(T,L)+(1-q)u_1(T,R)]+(1-p)[qu_1(B,L)+(1-q)u_1(B,R)]$

or $p[E_1(T,\alpha_2)]+(1-p)[E_1(B,\alpha_2)]$

Note regarding mixed strategies

- Thus, player 1's payoff is the weighted average of the expected payoffs from each action. So the expected payoff is linear in p.
- Example if $(E_1(T, \alpha_2) > E_1(B, \alpha_2)$. What does this tell us?
- Three possibilities: $p = 0$, $p = 1$ or, if $(E_1(T, \alpha_2) = E_1(B, \alpha_2)$, then any p will be a best response!
- What does this tell us? If we are looking for a mixed strategy eq. then we should ask the question "How should player 2 mix so that player 1 is indifferent between his actions?"

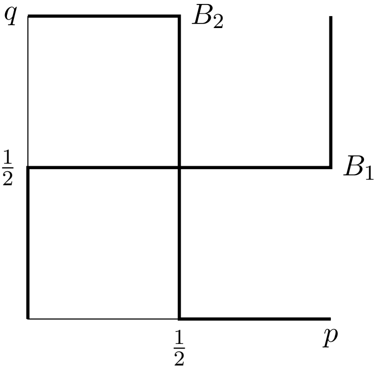
Back to Matching Pennies

What is P1's expected payoff from pure strategy $(p = 1)$ H if P2 plays q?

$q * 1 + (1 - q) * (-1) = 2q - 1$

What is P1's expected payoff from pure strategy T $(1 - p = 1)$ if P2 plays q?

$q * (-1) + (1 - q) * 1 = 1 - 2q$



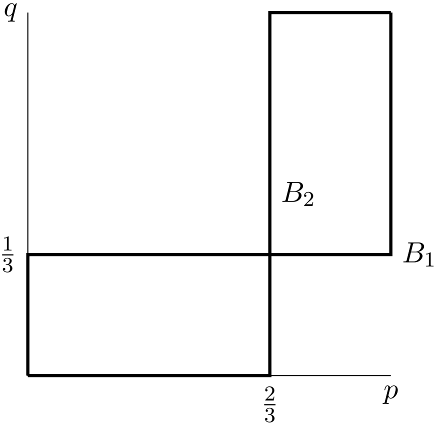
Best Responses for Matching Pennies Game

- Best response function for Player 1
  - $B_1(q) = 0$ if $q < \frac{1}{2}$
  - $B_1(q) = p : 0 \leq p \leq 1$ if $q = \frac{1}{2}$
  - $B_1(q) = 1$ if $q > \frac{1}{2}$
- Best response function for Player 2
  - $B_2(p) = 1$ if $p < \frac{1}{2}$
  - $B_2(p) = p : 0 \leq p \leq 1$ if $p = \frac{1}{2}$
  - $B_2(p) = 0$ if $p > \frac{1}{2}$

Battle of the Sexes revisited

		Actor 2	
		B	S
Actor 1	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

P1 expected payoff to B is $2 * q + 0 * (1 - q) = 2q$ and $0 * q + 1 * (1 - q) = 1 - q$ to S. Thus, P1 prefers B if $2q > 1 - q$ or $q > \frac{1}{3}$ and S if $q < \frac{1}{3}$



Best Responses for Battle of the Sexes Game

Alternatively, we can simply focus on when expected payoffs are equal:
$2q = 1 - q \rightarrow q = \frac{1}{3}$
Similarly for P2 (to make him indifferent): $E(B) = 1 * p + 0(1 - p)$ and $E(S) = 0 * p + 2 * (1 - p)$ and then $p = 2 - 2p \rightarrow p = \frac{2}{3}$.
Thus, there are three mixed strategy eq. in the game.

Best Responses for Battle of the Sexes Game

- Best response function for Player 1
    - $B_1(q) = 0$ if $q < \frac{1}{3}$
    - $B_1(q) = p : 0 \leq p \leq 1$ if $q = \frac{1}{3}$
    - $B_1(q) = 1$ if $q > \frac{1}{3}$
- Best response function for Player 2
    - $B_2(p) = 1$ if $p < \frac{1}{3}$
    - $B_2(p) = p : 0 \leq p \leq 1$ if $p = \frac{1}{3}$
    - $B_2(p) = 0$ if $p > \frac{1}{3}$

More general results

We can use the same ideas in larger games. Expected payoffs:

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) E_i(a_i, \alpha_{-i}) \tag{1}$$

		Actor 2		
		$Left(0)$	$Center(\frac{1}{3})$	$Right(\frac{2}{3})$
Actor 1	$Up(\frac{3}{4})$	$\cdot, 2$	$3, 3$	$1, 1$
	$Middle(0)$	$\cdot, \cdot$	$0, \cdot$	$2, \cdot$
	$Down(\frac{1}{4})$	$\cdot, 4$	$5, 1$	$0, 7$

Mixed Strategies

Proposition

A mixed strategy profile α^* *in a strategic game with vNM preferences in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player* i*,*

- *the expected payoff, given* α^*_{-i}*, to every action to which* α^*_i *assigns positive probability is the same*
- *the expected payoff, given* α^*_i*, to every action to which* α^*_i *assigns zero probability is at most the expected payoff to any action to which* α^*_i *assigns positive probability.*

Check: Is mixed strategy profile a mixed strategy Nash Equilibrium?

- For Player 1
  - $T = \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$
  - $M = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$
  - $B = \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 0 = \frac{5}{3}$
- For Player 2
  - $L = \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 4 = \frac{5}{2}$
  - $C = \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 1 = \frac{5}{2}$
  - $R = \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 7 = \frac{5}{2}$

Mixed Strategies

Proposition

Any strategic game with vNM preferences in which each player has a finite number of actions has a mixed strategy equilibrium

Definition

In . . . α_i **strictly dominates** action α_i' if

$$u_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for every } a_{-i}$$

		Actor 2	
		Left	*Right*
Actor 1	*Up*	1	1
	Middle	4	0
	Down	0	3

Mixed Strategies

Definition

In . . . α_i **weakly dominates** action α_i' if

$$u_i(\alpha_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for every } a_{-i} \tag{2}$$

and

$$u_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some } a_{-i} \tag{3}$$

An example

		Consumer	
		$Accept(q)$	$Reject(1-q)$
Expert	$Honest(p)$	$\pi, -rE-(1-r)I$	$(1-r)\pi, -rE'-(1-r)I$
	$Dishonest(1-p)$	$r\pi+(1-r)\pi', -E$	$0, -rE'-(1-r)I'$

Expert is indifferent if

$$q\pi + (1-q)(1-r)\pi = q(r\pi + (1-r)\pi')$$

or if

$$q = \frac{\pi}{\pi'}$$

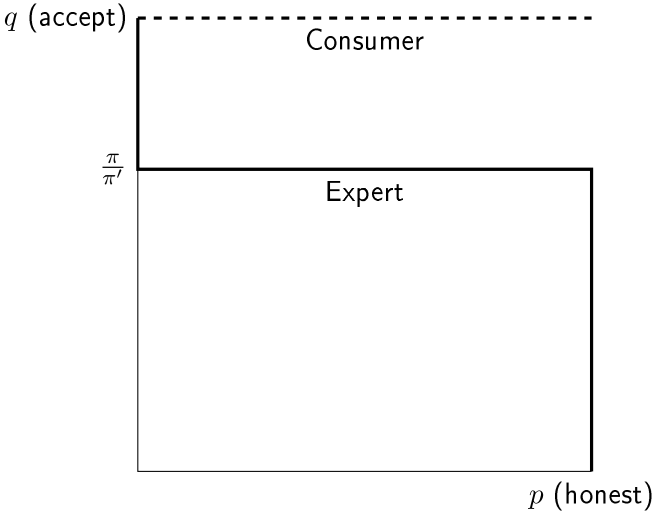
Consumer is indifferent if

$$p(-rE-(1-r)I)+(1-p)(-E) = p(-rE'-(1-r)I)+(1-p)(-rE'-(1-r)I')$$

or if

$$p = \frac{E-[rE'+(1-r)I']}{(1-r)(E-I')}$$

An example



An example