Formal Analysis: Lecture 2

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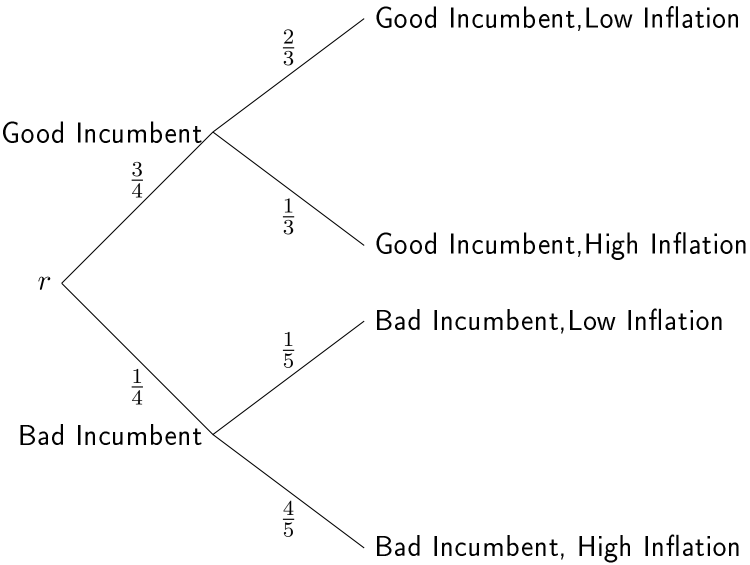
Learning and Bayes' Rule



Bayesian Reasoning

The likelihood the incumbent is good if we observe low inflation:

1. Agent knows that there is a $\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$ probability of reaching the top node.
2. And a $\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$ probability of reaching the third node.
3. After observing low inflation its 10 times as likely that the incumbent is good.
4. Let $p(l)$ be probability of good incumbent conditional on low inflation.
5. Because probabilities must sum to 1, $p(l) + \frac{p(l)}{10} = 1$ so that $p(l) = \frac{10}{11}$
6. $10p(l) + p(l) = 10$
7. $p(l)(10 + 1) = 10$
8. $p(l) = \frac{10}{11}$

Bayes' Rule

Let $A_1 ... A_N$ be disjoint events (i.e., no two can occur simultaneously) such that $\sum Pr(A_n) = 1$ and $Pr(A_n) > 0$ for all n. Let B be some other event. Then:

$$Pr(A_j|B) = \frac{Pr(B|A_j)Pr(A_j)}{\sum_{n=1}^{N} Pr(B|A_n)Pr(A_n)} \tag{1}$$

Bayes Incumbent/Inflation Example

Returning to our example, let A_1 be the event that the incumbent is good and A_2 be the event that she is bad. Event B is low inflation. The Bayes formulae are:

$$Pr(A_1|B) = \frac{Pr(B|A_1)Pr(A_1)}{Pr(B|A_l)Pr(A_l) + Pr(B|A_2)Pr(A_2)} \tag{2}$$

$$Pr(A_2|B) = \frac{Pr(B|A_2)Pr(A_2)}{Pr(B|A_1)Pr(A_1) + Pr(B|A_2)Pr(A_2)} \tag{3}$$

Bayes Incumbent/Inflation Example

- $Pr(A_1) = \frac{3}{4}$
- $Pr(A_2) = \frac{1}{4}$
- $Pr(B|A_1) = \frac{2}{3}$
- $Pr(B|A_2) = \frac{1}{5}$

$$Pr(A_1|B) = \frac{\frac{2}{3} \times \frac{3}{4}}{\frac{2}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{1}{4}} \tag{4}$$

and

$$Pr(A_2|B) = \frac{\frac{1}{5} \times \frac{1}{4}}{\frac{2}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{1}{4}} \tag{5}$$

Strategic/Normal form games

- A strategic game is defined as:
  - a set of players
  - a set of actions
  - and for each player, preferences over action profiles
- Alternatively, $\langle N, \{A_i, u_i(\cdot \ldots \cdot)\}_i \rangle$
  - N is set of actor $\{1, \ldots, n\}$
  - A_i is the set of actions available to actor i
  - $u_i(\cdot \ldots \cdot)$ is actor i's utility function (preferences), which (normally) depends on the actions of all the actors.

Strategic/Normal form games

- $a = (a_1, \ldots, a_n)$ denotes a vector of the actors' actions
- a_{-i} (or $a_{\sim i}$) denotes the actions of everyone except actor i
- then $a = (a_i, a_{-i})$

Nash Equilibrium

Consider a familiar game:

		Suspect 2	
		Quiet	*Fink*
Suspect 1	*Quiet*	$-4, -4$	$-25, -1$
	Fink	$-1, -25$	$-8, -8$

Here $N = \{1, 2\}, A_1 = A_2 = \{\text{Quiet}, \text{Fink}\}$ and, e.g., u_1(Fink, Quiet)=-1

Definition (Nash Equilibrium)

$$\forall i \in N, u_i(a^*) \geq u_i(a_i, a^*_{-i}), \forall a_i \in A_i \quad (6)$$

Dominated actions

Are there any actions such that we would always prefer taking some other action instead? Formally: action a_i'' dominates action a_i' if

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{7}$$

Consider this game:

		Actor 2		
		Left	*Center*	*Right*
	Up	$2,1$	$0,0$	$2,1$
Actor 1	*Middle*	$2,4$	$0,0$	$6,3$
	Down	$1,0$	$0,0$	$5,1$

Battle of the sexes/Bach or Stravinsky

Actor 2

Actor 1		*Bach*	*Stravinsky*
	Bach	$2,1$	$0,0$
	Stravinsky	$0,0$	$1,2$

Matching Pennies

Actor 2

Actor 1		$Heads$	$Tail$
	$Heads$	$1, -1$	$-1, 1$
	$Tails$	$-1, 1$	$1, -1$

HI-LO: Coordination

		Actor 2	
		Stag	*Hare*
Actor 1	*Stag*	2, 2	0, 1
	Hare	1, 0	1, 1

Best Response

Definition (Best Responses)

A players best response function (correspondence) specifies the the best response to given set of actions by the other players:

$$B_i(a_{-i}) = \{a_i \in A_i | u_i(a_i, a_{-i}) \geq u_(a_i', a_{-i}) \; \forall \; a_i' \in A_i\}$$

The best response is set valued (a correspondence).

Definition (Best Response Def of NE)

The action profile a^* is a NE iff

$$a_i^* \in B_i(a_{-i}^*), \; \forall i \in N$$

Best Response

		Actor 2		
		Left	*Center*	*Right*
	Up	$1, 2^*$	$2^*, 1$	$1^*, 0$
Actor 1	*Middle*	$2^*, 1^*$	$0, 1^*$	$0, 0$
	Down	$0, 1$	$0, 0$	$1^*, 2^*$

- What does actor 1 do if actor 2 is playing "Left"?
- What does actor 1 do if actor 2 is playing "Center"?
- etc. . .

Best Response and Nash Equilibria

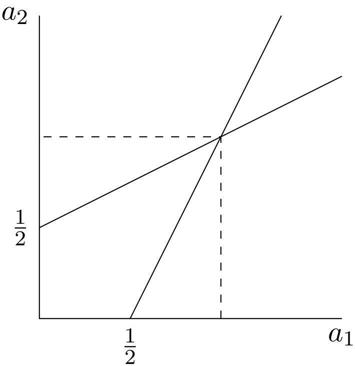
- $a_1^* = b_1(a_2^*)$
  - (M,L)
  - (T,C)
  - (T,R)
  - (T,B)
- $a_2^* = b_2(a_1^*)$
  - (T,L)
  - (M,L)
  - (M,C)
  - (B,R)
- (a_1^*, a_2^*)
  - (M,L)
  - (B,R)

Joint Project

Example 39.1

$\langle \{1,2\}, \{\mathbb{R}^+\}_i, \{a_i(c + a_j - a_i)\}_i \rangle$

That is, for any level that the other player picks my utility increases at first and then decreases. At zero when $a_i = 0$ and when $a_i = c + a_j$. We can show that the best response function is: $b_i(a_j) = \frac{1}{2}(c + a_j)$.



Solving for Nash Equilibria

- $a_1 = \frac{1}{2}(c + a_2)$
- $a_2 = \frac{1}{2}(c + a_1)$
- substituting we get $a_1 = \frac{1}{2}(c + \frac{1}{2}(c + a_1))$
- $a_1 = \frac{3}{4}c + \frac{1}{4}a_1$
- so that $a_1 = c$

Strictly Dominated Strategies

Strick domination

A strictly dominated action is not a best response to any actions of the other players: whatever the other players do, some other action is better. Are there any actions such that we would always prefer taking some other action instead? Formally: action a_i'' strictly dominates action a_i' if

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{8}$$

We say that a' is **strictly dominated**. A strictly dominated action is not used in any NE.

Strictly Dominated Strategies

Weak domination

Formally: In a strategic game with ordinal preferences, player i's action a_i'' **weakly dominates** her action a_i' if

$$u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{9}$$

and

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i} \tag{10}$$

for some list a_{-i} of the other players' actions.

We say that a' is **weakly dominated**.